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ACTION-ANGLE VARIABLES FOR THE HARMONIC OSCILLATOR: AMBIGUITY SPIN × DUPLICATION SPIN

by

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ACTION-ANGLE VARIABLES FOR THE HARMONIC OSCILLATOR:

AMBIGUITY SPIN × DUPLICATION SPIN

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ABSTRACT

The difficulties of obtaining for the harmonic oscillator a well defined unitary transformation to action-angle variables were overcome by M. Moshinsky and T.H. Seligman through the introduction of a spinlike variable (ambiguity spin) from a classical point of view. The difficulty of defining a unitary phase operator for the harmonic oscillator was overcome by Roger G. Newton also through the introduction of a spinlike variable (named duplication spin by us) but within a quantum framework. We investigate here the relation between the ambiguity spin and the duplication spin by introducing these two type of spins in the canonical transformation to action-angle variables. Doing this we are able to obtain both well defined unitary transformation and phase operator.

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1. INTRODUCTION

The use of action-angle variables for the description of physical systems is quite helpful specially in semiclassical analysis[(1)] but as is well knwon, the introduction of quantum action-angle variable for the harmonic oscillator (HO) hamiltonian presents some difficulties. The problem of determining the quantum unitary transformation[(2)] induced by the classical canonical transformation to action-angle variables was investigated by M. Moshinsky and P. Seligman[(3)]. The problem of the unitarity of the angle operator was investigated by R.G. Newton[(4)]. The root of the difficulties encountered in both problems lies in the lower boundedness[(5)] of the energy spectrum and for periodicity of the wave functions of the harmonic oscillator hamiltonian. In both works, the difficulties were overcome through the introduction of spinlike variables.

The ambiguity spins introduced by Moshinsky and Seligman[(3)] label different sheets of the original phase-space guaranteeing the bijectivity of the mapping of phase-spaces. One of the ambiguity spins corresponds to parity invariance of the canonical transformation and the other one corresponds to the $2\pi n$ translation invariance of the transformation.

The two-valued spin variable (it will be called here duplication spin) introduced by Newton[(4)] duplicates the Hilbert space, a vector in the enlarged space having two components, the upper component corresponding to the physical state. In the enlarged Hilbert space there exist a complete orthormal set of phase states so that the angle operator is unitary.

In the present work we investigate the relationship between the ambiguity spin, introduced by Moshinsky and Seligman[(3)]

within a classical framework, and the duplication spin, introduced by Newton[4] within a quantum framework. Studying the following canonical transformation to action-angle variables for the one-dimensional HO (we are using $\hbar=1$, $m=1$ and $\omega=1$)

$$J = \frac{1}{2}(p^2+q^2) \quad , \tag{1.1a}$$

$$\phi = \tan^{-1}\left(\frac{q}{p}\right) \quad , \tag{1.1b}$$

we show that the duplication spin plays a role equivalent to the parity ambiguity spin[3] with respect to the determination of the quantum unitary transformation induced by the classical canonical transformation (1). However, the duplication spin is absolutely necessary to guarantee the unitarity of the (quantum) phase operator $e^{i\phi}$. On the other hand, the introduction of the ambiguity spin related to the translation invariance of the phase variable ($\phi \to \phi + 2n\pi$) allows to obtain a complete and orthonormal set of phase states with $\phi \,\varepsilon\, (-\infty,\infty)$ and the restriction[4] of $\phi \,\varepsilon\, (-\pi,\pi)$ is removed. In section 2 we summarize the results obtained by R. Newton[4] and Moshinsky and Seligman[3].

In section 3 the duplication spin and the translation ambiguity spin are introduced for the canonical transformation (1) and the quantum unitary operator corresponding to this canonical transformation is obtained. In section 4 the quantum phase operator $e^{i\phi}$ is obtained using the Mello-Moshinsky transformation[6,3]. In section 5 the complete and orthonormal set of eigenstates of $e^{i\phi}$ is obtained. In section 6 the action J in ϕ-representation is shown to be $\left(-i\frac{\partial}{\partial\phi}\right)$ and the conclusions are summarized in section 7.

2. THE DUPLICATION SPIN[4] AND THE AMBIGUITY SPIN[3] - A BRIEF ACCOUNT

The introduction of the duplication spin (with components + and -) doubles the Hilbert space $\mathcal{H}$. A vector $||b\rangle$ of the enlarged Hilbert space $\hat{\mathcal{H}}$ will have two components, each one being a vector $|b\rangle$ of the original space $\mathcal{H}$, the upper component having spin + component and the lower spin -

$$||b\rangle = \begin{pmatrix} |b+\rangle \\ |b-\rangle \end{pmatrix} \quad . \tag{2.1}$$

The physical results are given by the upper component and are recovered by projection.

Given the eigenstates $|n\rangle$ of the HO hamiltonian, with eigenvalues $(n+\frac{1}{2})$, $n = 0, 1, \ldots$, the two component orthonormal basis $||n\rangle$ in the enlarged Hilbert space $\hat{\mathcal{H}}$ is given by

$$||n\rangle = \begin{cases} \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} \;, & \text{if} \quad n \geq 0 \\[2ex] \begin{pmatrix} 0 \\ |-n-1\rangle \end{pmatrix} \;, & \text{if} \quad n < 0 \end{cases} \tag{2.2}$$

where now $-\infty < n < \infty$ (integer).

The operators in $\hat{\mathcal{H}}$ (they will carry a caret) are 2×2 matrices which are given in terms of the operators in $\mathcal{H}$. The number operator $\hat{N}$, the raising and lowering operators, $\hat{E}^\dagger$ and $\hat{E}$, are defined as follows

$$\hat{N} = \begin{pmatrix} N & 0 \\ 0 & -N-1 \end{pmatrix} \quad , \quad \hat{E}^{\dagger} = \begin{pmatrix} E^{\dagger} & P_0 \\ 0 & E \end{pmatrix} \quad , \quad \hat{E} = \begin{pmatrix} E & 0 \\ P_0 & E^{\dagger} \end{pmatrix} \quad , \qquad (2.3)$$

where N, $E^{\dagger}$ and E are the corresponding operators in H,

$$N|n\rangle = n|n\rangle \ , E^{\dagger}|n\rangle = |n+1\rangle \ , \ E|n\rangle = (1-\delta_{n0})|n-1\rangle \quad , \quad n=0,1,2\ldots \qquad (2.4)$$

and P_0 is the projector on the HO ground-state,

$$P_0 = |0\rangle\langle 0| \quad .$$

It is easy to verify that

$$\hat{N}\,||n\rangle = n||n\rangle \quad , \quad \hat{E}^{\dagger}||n\rangle = ||n+1\rangle \quad , \text{ and } \hat{E}||n\rangle = ||n-1\rangle \quad , \quad n=0,\pm1,\pm2,\ldots \qquad (2.5)$$

so that $\hat{E}$ is a unitary operator even though the corresponding E is not.

Given any operator O in H, there exist at least two choices for the corresponding operator $\hat{O}$ in $\hat{H}$ which give identical physical results (upper component)

$$\hat{O} = O\,\hat{Q} = O \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \hat{O} = O\,\hat{1} = O \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \qquad (2.6)$$

For the HO hamiltonian the first alternative is the simplest one,

$$\hat{H} = H\hat{Q} = (\hat{N} + \tfrac{1}{2}) \quad , \qquad (2.7)$$

corresponding to eigenvalues $(n+\frac{1}{2})$, $n = 0,\pm1,\pm2,\ldots$, thus giving a spectrum that is not bounded from below (the other

alternative gives a spectrum doubly degenerate).

The presence of negative integers in the spectrum of the number operator $\hat{N}$ results in the existence of a unitary phase operator $e^{-i\phi}$ and of its canonically conjugate operator $-i\frac{\partial}{\partial\phi}$. The eigenstates $|\phi\rangle$ of the phase operator form a complete orthonormal set provided

$$\begin{aligned} &-\pi < \phi < \pi \quad , \\ &\langle\phi|\phi'\rangle = \delta(\phi-\phi') \quad , \\ &\int_{-\pi}^{\pi} d\phi\,|\phi\rangle\langle\phi| = \hat{1} \quad . \end{aligned} \qquad (2.8)$$

The ambiguity spin[3] is also introduced to enlarge the space. It is related to symmetry invariances of a canonical transformation and the labels of the irreducible representations of the symmetry group (ambiguity group) are the components of the ambiguity spin. As is well known[2] the quantum unitary transformation is not well defined when the corresponding operators do not have the same spectrum. This problem may be eliminated by the introduction of the ambiguity spin. In the particular case of the one-dimensional HO, Moshinsky and Seligman[3] defined the canonical transformation to action-angle variables by the equations

$$|J| = \frac{1}{2}(p^2+q^2) \quad , \qquad (2.9a)$$

$$\phi\,\frac{J}{|J|} = -\tan^{-1}\left(\frac{p}{q}\right) \quad , \qquad (2.9b)$$

and their inverse

$$\frac{1}{\sqrt{2}}(q-ip) = |J|^{1/2}\left\{\exp\left(i\,\phi\,\frac{J}{|J|}\right)\right\} \quad , \tag{2.10a}$$

$$\frac{1}{\sqrt{2}}(q+ip) = \exp\left\{-i\,\phi\,\frac{J}{|J|}\right\}|J|^{1/2} \quad . \tag{2.10b}$$

The above transformations remain invariant under translations $\phi \to \phi + 2\pi m$ (m arbitrary integer) and under the inversion $(\phi, J) \to (-\phi, -J)$. The ambiguity group is given by the semi-direct product of the abelian group T of translations by 2π with the inversion group I: $T \wedge I$. The irreducible representations of the ambiguity group $T \wedge I$ are characterized by the continuous parameter λ (translation spin), $0 \leq \lambda < 1$, and the discrete parameter $\sigma = \pm 1$ (parity spin). The translation and parity invariance of the transformation (2.9) results in the whole plane (q,p) being mapped on a strip between $\phi = 2\pi m$ and $\phi = 2\pi(m+1)$ on the upper half (ϕ,J) plane and again on the lower half (ϕ,J) plane. Considering then a double denumerable infinite sets of sheets of the (q,p) phase space the mapping $(q,p) \rightleftarrows (\phi,J)$ is bijective. Another way of considering this bijective mapping is to take a single sheet phase space (q,p) and many component functions labeled by the ambiguity spin λ and σ. Thus the introduction of λ and σ allows to obtain a unitary representation of the canonical transformation leading to action-angle variables for the HO. Correspondence between operators can also be obtained, the operator corresponding to $|J|$ having a continuum spectrum (as it should due to the bijectivity of the mapping). It is noted that in quantum mechanics the operator $\frac{1}{2}(p^2+q^2)$, associated to $|J|$ by the transformation, has the spectrum $n + \frac{1}{2}$, $n = 0,1,\ldots$.

Now, while the introduction of the duplication spin allows to obtain a unitary phase operator with eigenstates forming a complete orthonormal set for $-\pi < \phi < \pi$, it does not allow to obtain the unitary operator transforming to action-angle variables. On the other hand, the ambiguity spins provide a unitary operator transforming to action-angle variables but do not give a unitary phase operator. In order to get both things, we shall replace the parity spin by the duplication spin. And so we define the canonical transformation by (1.1) which does not have parity invariance and introduce the duplication spin.

3. THE UNITARY TRANSFORMATION FOR AMBIGUITY AND DUPLICATION SPINS

The canonical transformation (1.1) can be given by the following implicit equations

$$\frac{1}{(2)^{1/2}}(p+iq) = J^{1/2}\,e^{i\phi} \quad , \tag{3.1a}$$

$$\frac{1}{(2)^{1/2}}(p-iq) = e^{-i\phi}\,J^{1/2} \quad . \tag{3.1b}$$

The mapping of the phase-space (q,p) on the phase-space (ϕ,J) is not bijective for two reasons:

(i) Points (ϕ,J) on the lower half plane, $J < 0$, are not images of any point on the (q,p) plane.

(ii) All points $(\phi+2\pi m, J)$, m arbitrary integer, are mapped on the same point (q,p) (the transformation is invariant under the abelian group T of translations by 2π).

Distinction between the points on (q,p) corresponding to $(\phi+2\pi m, J)$ will be made by introducing the ambiguity spin[(3)]

as described in section 2. The inclusion of the lower-half (ϕ,J) plane will be achieved by duplicating the phase-space (q,p) via the introduction of the duplication spin[4] also described in section 2. The enlarged (q,p) space so obtained will be called intermediate space.

As we have seen, the ambiguity group T is characterized by the parameter λ restricted to the interval $0 \leq \lambda < 1$. Again, instead of considering the mapping of infinite denumberable sheets (q,p) on the upper half (ϕ,J) plane we shall consider a single sheet phase-space (q,p) and functions having infinite nondenumerable components labeled by λ. Now, the lower half plane, $J < 0$, is introduced via the duplication spin which doubles the phase-space (q,p). Each λ component will have two components characterized by the duplication spin components $+$ and $-$. The quantum final Hilbert space H associated with (ϕ,J) will have states with single component while the enlarged intermediate Hilbert space $\hat{H}$ will have states with nondenumerable infinite double components labeled by λ and by the duplication spin components $+$ or $-$. So, the basis given by (2.2) will be here the λ component of the two component orthonormal basis in the enlarged intermediate Hilbert space $\hat{H}$. This two-component λ component will be denoted by $||n>_\lambda$, i.e.,

$$||n>_\lambda = \begin{cases} \begin{pmatrix} |n> \\ 0 \end{pmatrix}_\lambda , & \text{if} \quad n \geq 0 \\ \begin{pmatrix} 0 \\ |-n-1> \end{pmatrix}_\lambda , & \text{if} \quad n < 0 , \end{cases} \tag{3.2}$$

$-\infty < n < \infty$, integer, $0 \leq \lambda < 1$.

Following the procedure of Moshinsky and Seligman[3] we consider now the complete set (orthonormal) of states in the final Hilbert space H

$$\frac{1}{(2\pi)^{1/2}} \exp(i\nu q") ,$$

(as in ref. (3) we shall adopt the convention that variables carrying a double prime refer to the final Hilbert space) and write the real variable ν in terms of the ambiguity spin λ,

$$\nu = n+\lambda , \quad n = 0,\pm1,\pm2,\ldots \quad \text{and} \quad 0 \leq \lambda < 1 ,$$

so that for a fixed λ, the functions $(2\pi)^{1/2} \exp(i(n+\lambda)q")$ provide a basis for the irreducible representations of the ambiguity group T[3]. In the momentum representation, the corresponding set, $\delta(p"-\nu)$, will be decomposed is subsets

$$\delta(p" - (n+\lambda)) , \quad n = 0,\pm1,\ldots \quad \text{and} \quad 0 \leq \lambda < 1 . \tag{3.3}$$

And the unitary representation of the canonical transformation (3.1) (or (1.1)) are obtained from the basis (3.3) for the irreducible representations of the ambiguity group T. The subset $\delta(p" - (n+\lambda))$, $n = 0,\pm1,\ldots$, for a given λ, must be mapped onto the corresponding component $||n>_\lambda$ (3.2) in the enlarged intermediate Hilbert space $\hat{H}$. In the momentum representation $||n>$ is given by

$$\hat{\psi}_n^\lambda(p') = \begin{cases} \begin{pmatrix} \psi_n(p') \\ 0 \end{pmatrix}_\lambda , & \text{if} \quad n \geq 0 \\ \begin{pmatrix} 0 \\ \psi_{-n-1}(p') \end{pmatrix}_\lambda , & \text{if} \quad n < 0 , \end{cases} \tag{3.4}$$

where $\psi_n(p') = \langle p'|n\rangle$ are the eigenfunctions of the HO hamiltonian therefore satisfying the equation

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial p'^2} + \frac{1}{2}p'^2\right)\psi_n(p') = \left(n+\frac{1}{2}\right)\psi_n(p') \quad , \; n \text{ integer} \geq 0 \quad . \qquad (3.5)$$

So, for each λ value, the eigenfunctions of the momentum, $\delta(p''-(n+\lambda))$, are mapped onto the eigenfunctions $\tilde{\psi}_n^\lambda(p')$ of the HO hamiltonian, and the matrix elements $\langle p'|U^\lambda|p''\rangle$ of the unitary transformation corresponding to the canonical transformation (3.1) are given by

$$\langle p'|U^\lambda|p''\rangle = \sum_{n=-\infty}^{\infty} \tilde{\psi}_n^\lambda(p')\,\delta(p''-(n+\lambda)) \quad . \qquad (3.6)$$

For the full unitary operator U we propose the following formal expression

$$U = \int_0^1 d\lambda\; U^\lambda\, B^\lambda \quad , \qquad (3.7)$$

where B^λ is a projector that acting on an arbitrary state of the final Hilbert space selects that part associated with λ . It should be remarked that the unitarity condition take the following form,

$$\begin{aligned} \langle p'|U^\dagger U|p''\rangle &= \delta(p'-p'') \quad , \\ \langle p'|UU^\dagger|p''\rangle &= \delta(p'-p'')\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ||\,\delta(\lambda-\lambda')\,|| \quad . \end{aligned} \qquad (3.8)$$

4. CORRESPONDENCE OF OPERATORS IN THE (ENLARGED) INTERMEDIATE AND FINAL HILBERT SPACES

The correspondence of operators in the enlarged intermediate Hilbert space $\tilde{H}$ and operators in the final Hilbert space is obtained by using the Mello-Moshinsky (MM) equations[3,6]. These equations consist of differential equations for the unitary representation of the operator U associated to a canonical transformation. Any classical canonical transformation can be given by implicit equations,

$$\begin{aligned} F(p,q) &= \mathcal{F}(Q,P) \\ G(q,p) &= \mathcal{G}(Q,P) \end{aligned} \qquad (4.1)$$

with the following condition on the Poisson brackets

$$\{F,G\}_{q,p} = \{\mathcal{F},\mathcal{G}\}_{Q,P} \quad .$$

The unitary operator U that performs the transformation of operators q,p onto operators Q,P is given by

$$Q = U\,q\,U^{-1} \quad , \quad P = U\,p\,U^{-1} \quad . \qquad (4.2)$$

The above equation is well defined[2] only when operators q,p and operators Q,P have the same spectra, which classically means that the mapping of the phase-spaces is bijective.

For any function that is a power series of p,q , it follows from (4.2) that

$$U\,K(q,p)\,U^{-1} = K(Q,P)$$

or,

$$K(Q,P)\,U = U\,K(q,p)$$

Particularly for $F(q,p)$ and $G(q,p)$ (given in (4.1)) equation (4.2) reads

$$F(q,p)\,U = U\,F(q,p)$$

$$G(q,p)\,U = U\,G(q,p) \quad ,$$

which in the momentum representation constitute the MM equations[6,3]

$$\begin{aligned} F(i\frac{\partial}{\partial p'}, p')\,\langle p'|U|p''\rangle &= \left[F^{\dagger}(i\frac{\partial}{\partial p''}, p'')\right]^{*} \langle p'|U|p''\rangle \quad , \\ G(i\frac{\partial}{\partial p'}, p')\,\langle p'|U|p''\rangle &= \left[G^{\dagger}(i\frac{\partial}{\partial p''}, p'')\right]^{*} \langle p'|U|p''\rangle \quad . \end{aligned} \tag{4.3}$$

For each λ component there are raising and lowering operators, $\hat{E}_{\lambda}^{\dagger}$ and $\hat{E}_{\lambda}$ given by (2.3). The raising and lowering operator for the whole intermediate space $\hat{H}$ can be written formally as $\hat{E}^{\dagger} = \hat{E}_{\lambda}^{\dagger}\,||\delta(\lambda-\lambda')||$ and $\hat{E} = \hat{E}_{\lambda}\,||\delta(\lambda-\lambda')||$, respectively. We shall now obtain the operator, in the final Hilbert space, corresponding to these raising and lowering operators in the intermediate Hilbert space (we shall do the explicit derivation for $\hat{E}^{\dagger}$ only).

Taking initially a single λ component, we apply $\hat{E}_{\lambda}^{\dagger}$ to both sides of equation (3.6) (unitary representation of U^{λ}) obtaining

$$\hat{E}_{\lambda}^{\dagger}\langle p'|U^{\lambda}|p''\rangle = \sum_{n=-\infty}^{\infty} \hat{E}^{\dagger}\,\hat{\psi}_n(p')\;\delta(p''-(n+\lambda)) \equiv \sum_{n=-\infty}^{\infty} \psi_{n+1}^{\lambda}(p')\;\delta(p''-(n+\lambda)) =$$

$$= \sum_{n=-\infty}^{\infty} \hat{\psi}_n^{\lambda}(p')\;\delta(p''+1-(n+\lambda)) =$$

$$= e^{\frac{\partial}{\partial p''}} \sum_{n=-\infty}^{\infty} \hat{\psi}_n^{\lambda}(p')\,\delta(p''-(n+\lambda)) = e^{\frac{\partial}{\partial p''}} \langle p'|U^{\lambda}|p''\rangle \quad , \tag{4.4}$$

where we have made use of the property that the hyperdifferential operator $\exp\left(\frac{\partial}{\partial p''}\right)$ changes p'' into $p''+1$.

Writing equation (4.4) in the form

$$\hat{E}_{\lambda}^{\dagger}\langle p'|U^{\lambda}|p''\rangle = \left\{\left(e^{-\frac{\partial}{\partial p''}}\right)^{\dagger}\right\}^{*} \langle p'|U^{\lambda}|p''\rangle \quad ,$$

and comparing with the MM equations (4.3) (remembering that p'' is the momentum in the final space, i.e., J) we see that the operator $\hat{E}_{\lambda}^{\dagger}$ corresponds to $e^{i\phi}$. Therefore, making use of the projector B^{λ}, we have

$$\hat{E}^{\dagger} \rightleftarrows \int_0^1 d\lambda\; e^{i\phi}\,B^{\lambda} = e^{i\phi} \quad . \tag{4.5a}$$

Analogously,

$$\hat{E} \rightleftarrows e^{-i\phi} \quad . \tag{4.5b}$$

For each λ value the HO hamiltonian in the intermediate space $\hat{H}$ is $\hat{H}_{\lambda} = (\hat{N}_{\lambda} + \frac{1}{2})$ ($\hat{N}$ given by (2.3)) and the full HO hamiltonian operator in the intermediate space is $\hat{H} = \hat{H}_{\lambda}\,||\delta(\lambda-\lambda')||$. Applying $\hat{H}_{\lambda}$ on both sides of equation (3.6) we get

$$\hat{H}_{\lambda}\langle p'|U^{\lambda}|p''\rangle = \sum_{n=-\infty}^{\infty} (n+\frac{1}{2})\,\hat{\psi}_n^{\lambda}(p')\,\delta(p''-(n+\lambda)) = (p''+\frac{1}{2}-\lambda)\langle p'|U^{\lambda}|p''\rangle \;,$$

so that

$$\tilde{H}_\lambda \rightleftarrows (J+\frac{1}{2}-\lambda) \quad ,$$

therefore

$$\tilde{H} \rightleftarrows \int_0^1 d\lambda \ (J+\frac{1}{2}-\lambda) \ B^\lambda \quad . \tag{4.6}$$

It can easily be verified that the operator in the right hand side of (4.6) has spectrum $(n+\frac{1}{2})$, $n=0,\pm1,\pm2,\ldots$.

We shall now use the same procedure to find the operator in the intermediate enlarged Hilbert space corresponding to the action J (given by p''). Applying p'' on both sides of (3.6) we get

$$p''\langle p'|U^\lambda|p''\rangle = \sum_{n=-\infty}^{\infty} \tilde{\psi}_n^\lambda(p')p''\delta(p''-(n+\lambda)) =$$

$$= \sum_{n=-\infty}^{\infty} (n+\lambda)\tilde{\psi}_n^\lambda(p')\delta(p''-(n+\lambda)) \quad ,$$

therefore

$$\int_0^1 d\lambda \ J = J \rightleftarrows ||(\tilde{N}_\lambda+\lambda)\delta(\lambda-\lambda')|| \quad , \tag{4.7}$$

so that the operator $||(\tilde{N}_\lambda+\lambda)\delta(\lambda-\lambda')||$ has a continuum spectrum like the action J (as it should in order to have a well defined operator U).

5. COMPLETENESS RELATION FOR THE EIGENSTATES OF $e^{i\phi}$

Let $|\phi'\rangle$ be eigenstate of $e^{i\phi}$,

$$e^{i\phi}|\phi'\rangle = e^{i\phi'}|\phi'\rangle \quad . \tag{5.1}$$

The eigenstates $|p''\rangle$ of the action J can be written as

$$|p''\rangle = |n+\lambda\rangle \quad , \quad 0\le\lambda<1 \quad , \quad n=0,\pm1,\pm2,\ldots , \tag{5.2}$$

for which the orthogonality takes the form

$$\langle n+\lambda|n'+\lambda'\rangle = \delta_{nn'}\,\delta(\lambda-\lambda') \quad . \tag{5.3}$$

The eigenstates $|\phi'\rangle$ can be expanded in terms of the states (5.2)

$$|\phi'\rangle = \frac{1}{(2\pi)^{1/2}} \int_0^1 d\lambda \sum_{n=-\infty}^{\infty} e^{-i(n+\lambda)\phi'}|n+\lambda\rangle \quad . \tag{5.4}$$

Using this expansion (5.4) for eigenstates $|\phi'\rangle$ we shall obtain the orthogonality relation

$$\langle\phi'|\phi''\rangle = \frac{1}{2\pi}\int_0^1 d\lambda \int_0^1 d\lambda' \sum_{n,n'=-\infty}^{\infty} \exp\left\{i\left[(n+\lambda)\phi'-(n'+\lambda')\phi''\right]\right\} \langle n+\lambda|n'+\lambda'\rangle =$$

$$= \frac{1}{2\pi}\int_0^1 d\lambda \sum_{n=-\infty}^{\infty} \exp\left\{in(\phi'-\phi'') + i\lambda(\phi'-\phi'')\right\}$$

where we have used the orthogonality (5.3). Using the identity (Poisson formula)[7]

$$\sum_{n=-\infty}^{\infty} \exp\left[in(\phi'-\phi'')\right] = \sum_{n=-\infty}^{\infty} \delta(\phi'-\phi''-2n\pi) \quad ,$$

we get

$$\langle\phi'|\phi''\rangle = \frac{1}{2\pi}\int_0^1 d\lambda\, \exp\left[i\lambda(\phi'-\phi'')\right] \sum_{n=-\infty}^{\infty} \delta(\phi'-\phi''-2n\pi) =$$

$$= \frac{e^{i(\phi'-\phi'')}-1}{i(\phi'-\phi'')} \sum_{n=-\infty}^{\infty} \delta(\phi'-\phi''-2n\pi) \quad .$$

Now, as

$$\lim_{x\to 2n\pi} \frac{e^{ix}-1}{ix} = \delta_{n0} \quad ,$$

it follows the orthogonality

$$\langle\phi'|\phi''\rangle = \delta(\phi'-\phi'') \quad , \tag{5.5}$$

without any restriction on the range of ϕ' or ϕ''.

To obtain the completeness relation we first calculate $\langle|\phi'\rangle$ where $|\rangle$ is an arbitrary state in the final Hilbert space,

$$\langle|\phi'\rangle = \int_0^1 d\lambda \sum_{n=-\infty}^{\infty} \langle|n+\lambda\rangle\langle n+\lambda|\phi'\rangle = \int_0^1 d\lambda \sum_{n=-\infty}^{\infty} \langle|n+\lambda\rangle \frac{1}{(2\pi)^{1/2}} \exp\left[-i(n+\lambda)\phi'\right] .$$

This relation can be inverted giving

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \exp\left[+i(n'+\lambda')\phi'\right] \langle|\phi'\rangle\, d\phi' = \langle|n'+\lambda'\rangle \quad ,$$

which is the same as

$$\langle|n'+\lambda'\rangle = \int_{-\infty}^{\infty} d\phi' \,\langle|\phi'\rangle\langle\phi'|n'+\lambda'\rangle$$

so that formally

$$\int_{-\infty}^{\infty} d\phi' \; |\phi'\rangle\langle\phi'| = 1 \quad . \tag{5.6}$$

(Notice the range of integration, $(-\infty,\infty)$, instead of $(-\pi,+\pi)$ in (2.8)).

6. REPRESENTATIONS ϕ AND J

In deriving the correspondence between operators in the intermediate enlarged Hilbert space and the final Hilbert space we have assumed J to be diagonal and in this representation $\phi = i\frac{\partial}{\partial J}$. We shall first show that $J = -i\frac{\partial}{\partial\phi}$ in a representation in which $e^{-i\phi}$ is diagonal (we follow Newton's procedure in ref. 4). We have

$$\left[e^{-i\phi}, J\right] = e^{-i\phi} \quad ,$$

or, more generally,

$$\left[e^{-i\phi} \;,\; J+h(e^{-i\phi})\right] = e^{-i\phi} \quad , \tag{6.1}$$

where h is an arbitrary function of $e^{-i\phi}$.

Equation (6.1) in the representation ϕ' given in (5.4) is

$$\langle\phi'|e^{-i\phi}\,(J+h(e^{-i\phi})) - (J+h(e^{-i\phi}))\,e^{-i\phi}|\phi''\rangle = \langle\phi'|e^{-i\phi}|\phi''\rangle$$

from which it follows that

$$\langle\phi'|(J+h(e^{-i\phi}))|\phi''\rangle = \frac{e^{-i\phi'}}{e^{-i\phi'}-e^{-i\phi''}}\ \delta(\phi'-\phi'') \equiv \frac{\delta(\phi'-\phi'')}{i(\phi'-\phi'')} = -i\frac{\partial}{\partial\phi'}\delta(\phi'-\phi'')\ ,$$

or,

$$\langle\phi'|J|\phi''\rangle = \left\{-i\frac{\partial}{\partial\phi'} + h(e^{-i\phi'})\right\}\delta(\phi'-\phi'')\ . \tag{6.2}$$

Now, using the basis that diagonalises J (in form (5.2)), we have

$$\langle\phi'|J|n+\lambda\rangle = (n+\lambda)\ \langle\phi'|n+\lambda\rangle = (n+\lambda)\ \frac{1}{(2\pi)^{1/2}}\ e^{i(n+\lambda)\phi'}\ . \tag{6.3}$$

However, using (6.2) and the completeness relation (5.6) we get

$$\langle\phi'|J|n+\lambda\rangle = \int_{-\infty}^{\infty} d\phi''\ \langle\phi'|J|\phi''\rangle\langle\phi''|n+\lambda\rangle = \frac{1}{(2\pi)^{1/2}}\left\{-i\frac{\partial}{\partial\phi'} + h(e^{-i\phi'})\right\}e^{i(n+\lambda)\phi'} =$$

$$= \frac{(n+\lambda)}{(2\pi)^{1/2}}\ e^{i(n+\lambda)\phi'} + \frac{h(e^{-i\phi'})}{(2\pi)^{1/2}}\ e^{i(n+\lambda)\phi'}\ . \tag{6.4}$$

Comparison of (6.3) and (6.4) gives $h=0$ so that in ϕ representation

$$J = -i\frac{\partial}{\partial\phi}\ . \tag{6.5}$$

As the action-angle (final) Hilbert space has the same structure of the position-momentum Hilbert space, it follows[2] that

$$\phi = i\frac{\partial}{\partial J}\ . \tag{6.6}$$

7. CONCLUSIONS

We have enlarged the HO Hilbert space by introducing the ambiguity spin λ (associated to the abelian group T of translations by 2π) and the duplication spin (which allows the definition of a number operator for the HO that has integral eigenvalues of both signs). As a result, we have been able to obtain a well defined quantum unitary operator leading to action-angle variables and a unitary phase operator $e^{-i\phi}$. The eigenstates of the phase operator form a complete orthonormal set with $-\infty < \phi < \infty$ ((5.5) and (5.6)) the restriction $-\pi < \phi < \pi$ present in (2.8) being eliminated. As in reference 3, the operator corresponding to the action J has a continuum spectrum (4.7) but at the same time, the operator $\frac{p^2+q^2}{2}$, associated to J via the transformation (1.1), has spectrum $(n+\frac{1}{2})$, $n = 0,\pm1,\pm2,\ldots$.

So, the duplication spin combined with the ambiguity spin allows to obtain all the features of each one of them plus the elimination of the restriction $-\pi < \phi < \pi$.

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