UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-491

BOUND-STATES OF N PARTICLES: A VARIATIONAL APPROACH

by

J. Fernando Perez, C.P. Malta and F.A.B. Coutinho
Instituto de Física, Universidade de São Paulo

Setembro/1984

"BOUND-STATES OF N PARTICLES: A VARIATIONAL APPROACH"

J. Fernando Perez[†], C.P. Malta and F.A.B. Coutinho[†]

Depto. de Física Matemática, Instituto de Física da Universidade de São Paulo, CP 20516, 01000 São Paulo, SP, Brazil

[†]Partially supported by CNPq.

Number of copies submitted: 2

ABSTRACT

Using a variational technique we provide sufficient conditions for the existence of a bound-state in a system of N particles in one-, two- and three-dimensions.

1. INTRODUCTION

In this paper we explore the variational method for obtaining sufficient conditions for the existence of a bound-state in a system of N particles in ν-dimension iteracting via two-body potentials $V_{ij}(\vec{r}_i-\vec{r}_j)$.

Let H_N denote the Hamiltonian of the N-particle system (with center of mass (CM) kinetic energy removed) and ε_N the energy of its continuum threshold. If we can find a wave function Φ_N such that

$$(\Phi_N, H_N \Phi_N) < \varepsilon_N (\Phi_N, \Phi_N) \quad , \tag{1}$$

the variational principle guarantees, then, the existence of at least one bound-state of N particles (below the continuum).

The difficulty associated with the method lies in the determination of ε_N. For locally square-integrable two-body potentials vanishing at infinity, ε_N is given by Hunziker's theorem [1,2] which envolves the knowledge of the bound-states of all subsystems of the whole system. Denoting by C a cluster, $C \equiv \{i_1, \ldots, i_{n(C)}\} \subset \{1, \ldots, N\}$, and by H^C the Hamiltonian of the subsystem formed by C, after the removal of CM kinetic energy, and by E_0^C the energy of the infimum of the spectrum of H^C then Hunziker's theorem [1,2] gives

$$\varepsilon_N = \inf_{\substack{\{C_1,\ldots,C_\ell\} \\ \bigcup_{i=1}^{\ell} C_i = \{1,\ldots,N\} \\ C_i \cap C_j = \phi\ ,\ i\neq j \\ 1\leq \ell \leq N}} \sum_{i=1}^{\ell} E_0^{C_i} \quad , \tag{2}$$

where the infimum is taken over all possible decomposition of $\{1,\ldots,N\}$ into disjoint clusters.

For the two-body problem $\varepsilon_2 = 0$ and therefore it is not difficult to find sufficient conditions on the potential for the existence of bound-states. If $N \geq 3$, however, $\varepsilon_N < 0$ in general thus making the problem not so simple. To bypass this difficulty we use a recursive procedure on the particle number N to show that there exist simple sufficient conditions on the two-body potentials that ensure the existence of bound-states for arbitrary N in ν-dimension ($\nu = 1$, 2 or 3).

The physical idea behind our method is dimension dependent. In three-dimension it is basically a "two-body mechanism": for a certain class (to be made precise below) of purely attractive two-body interactions, given two <u>bound</u> clusters $C_1(n(C_1)=N_1)$ and $C_2(n(C_2)=N_2)$, it is possible to bind them together provided there is at least one pair of particles each one in a different cluster which can form a bound-state. On the other hand, in one- and two-dimensions it is a "two-cluster mechanism": for "globally attractive" two-body potentials, i.e $\int V_{ij}\, d^\nu x < 0$, given two <u>bound</u> clusters

C_1 and C_2, it is possible to bind them together since the "effective" inter-cluster potential also satisfies $\int V^{eff}_{C_1C_2}\, d^\nu x < 0$. The difference between $\nu = 1, 2$ and $\nu=3$ cases has its origin in the fact that in one- and two-dimensions a two-body interaction with $\int V\, d^\nu x < 0$ always binds [3,4] two particles but in three-dimension this is not the case even if V is purely attractive. However, if the particles are identical (bosons or fermions) we can also show that a N-body system will exhibit bound-states even if the two-body system has no bound-states, provided N is big enough (along a subsequence). In fact, large particle number favours the existence of bound-states of identical particles (bosons or fermions): classically catastrophic potentials (for instance, globally attractive potentials or potentials with an attractive core) [5] remain so in the quantum case, i.e. $\lim_{N\to\infty} \frac{-E_N}{N} = \infty$. Based on that, we show that for these potentials there exists an infinite sequence $2 \le N_0 \le N_1 \le \ldots$ such that H_{N_i} has at least one bound-state.

This paper is organized as follows: in section 2 we derive useful sufficient condition for a two-particle system to have a bound-state with energy below $-\alpha^2$ in $\nu = 1, 2$ or 3 dimensions. For $\nu = 1$ or 2 and $\alpha = 0$ we recover the above mentioned result that if the potential satisfies $\int V\, d^\nu x < 0$ the system has always a bound-state. For $\nu = 3$, and $\alpha = 0$, we obtain a sufficient condition which is simpler than that obtained in [5]; also, if the potential has spherical symmetry we recover Calogero's "best" sufficient condition [7]. We also

derive sufficient conditions for the existence of a bound-state of a given angular momentum. Moreover, we show that some of the sufficient conditions provided by Calogero [7] are improved by the variational approach.

In section 3 we derive sufficient conditions for the existence of a bound-state in the N-body problem in one-, two- and three-dimensions. Part of these results have been announced in [3,8].

In section 4 we prove our results for large number of identical particles.

In the Appendix we collect some kinematical facts for N-body systems which are used in this work.

2. BOUND-STATES IN THE TWO-BODY PROBLEM

A collection of sufficient conditions on V for the existence of bound-states of H_2 with energy below $-\alpha^2$ is obtained by varying the trial function in the inequality (1) (with $\varepsilon_2 = -\alpha^2$)

$$(\phi^\alpha_R, H_2\, \phi^\alpha_R) < -\alpha^2 (\phi^\alpha_R, \phi^\alpha_R) \quad . \tag{3}$$

Particularly useful sufficient conditions are those expressed in terms of simple integrals of the two-body potential V [6]. For instance, very simple conditions are obtained by taking as trial a function ϕ_R^α that at short range behaves as $e^{-\alpha r}$ and at long range behaves as the solution of the free equation with energy $-\alpha^2$:

$$\phi_R^\alpha(r) = e^{-\alpha r}\,\phi(r) \qquad \text{for} \quad r < R$$
$$\phi_R^\alpha(r) = e^{-\alpha R}\,\phi(r)\,\frac{H(\alpha r)}{H(\alpha R)} \qquad \text{for} \quad r > R \tag{4}$$

$(r = |\vec{r}|)$, where $\phi(r)$ is arbitrary and $H(\alpha r)$ is the solution of the modified Helmholtz equation,

$$(-\Delta+\alpha^2)\,H(\alpha r) = 0 \; . \tag{5}$$

(We are taking energy in units of $\hbar^2/2\mu$, μ being the reduced mass of the two particles).

So, in one-dimension we take

$$\phi_R^\alpha(x) = e^{-\alpha|x-x_0|}\,\phi_R(x-x_0) \tag{6}$$

where x_0 is arbitrary and $\phi_R(x-x_0) \in L^2(\mathbb{R}^1)$ is such that for $|x-x_0| < R$, $\phi_R(x-x_0) = 1$ and for $|x-x_0| > R$, $\phi_R(x-x_0)$ starts at 1 and $\phi_R(x-x_0) \to 0$ as $|x-x_0| \to \infty$. By making the scaling $\phi_R(x-x_0) \to \beta^{1/2}\,\phi_R(\beta(x-x_0))$ and letting $\beta \to 0$ we obtain the condition

$$2\alpha + \int_{-\infty}^{\infty} e^{-2\alpha|x-x_0|}\,V(x)\,dx < 0 \; .$$

Setting $\alpha = 0$ we recover the known result [3,4] that in one-dimension a globally attractive potential, i.e. $\int V\,dx < 0$, always possesses at least one bound-state.

In two-dimension we take

$$\phi_R^\alpha(r) = e^{-\alpha r} \qquad \text{for} \quad r < R \; ,$$
$$\phi_R^\alpha(r) = \frac{e^{-\alpha R}}{K_0(\alpha R)}\,K_0(\alpha r) \qquad \text{for} \quad r > R \; , \tag{7}$$

where $K_0(\alpha r)$ is the modified Bessel function, and obtain the following sufficient condition for the existence of a bound-state of energy less than $-\alpha^2$

$$-\frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^R e^{-2\alpha r}\,V(r,\theta)\,r\,dr - \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_R^\infty e^{-2\alpha R}\,\frac{K_0(\alpha r)}{K_0(\alpha R)}\,V(r,\theta)\,r\,dr \geq$$
$$\geq \left(\frac{1-e^{-2\alpha R}}{2}\right) - \alpha R\,e^{-2\alpha R}\left\{1 + \frac{K_0'(\alpha R)}{K_0(\alpha R)}\right\} \; , \tag{8}$$

where $K_0'(\alpha R) = \left.\dfrac{dK_0(\alpha r)}{d(\alpha r)}\right|_{\alpha r=\alpha R}$.

Again, setting $\alpha = 0$ we recover the result [3,4] that in two-dimension a globally attractive potential ($\int V\,d^2x < 0$) always possesses at least one bound-state.

Finally, in three-dimension we take

$$\Phi_R^\alpha(r) = \frac{1}{R^{1/2}} \; e^{-\alpha r} \qquad \text{for} \quad r < R \;,$$
$$\Phi_R^\alpha(r) = R^{1/2} \; \frac{e^{-\alpha r}}{r} \qquad \text{for} \quad r > R \;, \tag{9}$$

obtaining the following sufficient condition for the existence of a bound-state of energy less than $-\alpha^2$

$$-\frac{1}{4\pi}\int d\Omega \int_0^R \frac{e^{-2\alpha R}}{R} \; V(r,\Omega)\, r^2 dr - \frac{1}{4\pi}\int d\Omega \int_R^\infty R \,\frac{e^{-2\alpha r}}{r^2} \;\; V(r,\Omega)\, r^2\, dr \;\geq$$
$$\geq \; \frac{1-e^{-2\alpha R}}{2\alpha R} \;. \tag{10}$$

Setting $\alpha = 0$ obtains

$$-\frac{1}{4\pi}\int d\Omega \int_0^R \frac{1}{R}\, V(r,\Omega)\; r^2\, dr - \frac{1}{4\pi}\int d\Omega \int_R^\infty R\, V(r,\Omega)\, dr \;\geq\; 1 \;, \tag{11}$$

which is simpler than the condition obtained by Chadan and Martin [6]. In the particular case of spherically symmetric potential, condition (11) reduces to Calogero's [7] "best" sufficient condition.

We shall now present the variational version of the other sufficient conditions ($\alpha=0$) derived by Calogero in reference 7. Taking as trial function $\Phi_R(r) = R^{1/2}/(r+R)$, a sufficient condition for a spherically symmetric potential to hold a bound-state is

$$\int_0^\infty \frac{R}{(r+R)^2}\, V(r)\, r^2\, dr \;<\; -\frac{1}{3} \;. \tag{12}$$

Taking $\Phi_R(r) = (R^{1/2}/r)(1-e^{-r/R})$ as trial function we get the condition

$$\int_0^\infty \frac{R}{r^2}\, (1-e^{-r/R})^2\, V(r)\, r^2\, dr \;<\; -\frac{1}{2} \;. \tag{13}$$

Conditions (12) and (13) should be compared with Calogero's [7] conditions (3.15) and (3.17) respectively. In both cases the variational method produced improvement.

<u>Remark</u> – The fact that for $\alpha = 0$ the trial function is not square integrable is of no importance: if $\lim_{\alpha\to 0} (\Phi_R^\alpha, H_2\, \Phi_R^\alpha) < 0$ for a sequence Φ_R^α, then, for α sufficiently small $(\Phi_R^\alpha, H_2\, \Phi_R^\alpha) < 0$ and the variational principle guarantees the existence of a bound-state.

For spherically symmetric potentials, our method can be readily adapted to provide sufficient conditions for the existence of a bound-state having energy less than $-\alpha^2$ and of a given angular momentum ℓ. Our recipe in $\nu = 3$ is then to use as trial function the regular and irregular solutions of the modified Bessel equation,

$$-\frac{d^2\chi_\ell}{dr^2} - \frac{2}{r}\frac{d\chi_\ell}{dr} + \frac{\ell(\ell+1)}{r^2}\,\chi_\ell + \alpha^2\chi_\ell = 0 \;, \tag{14}$$

matched at an arbitrary point $r = R$. Thus, the trial function is

$$\phi_R^\ell(r) = \frac{(\alpha R)^{1/2}}{K_{\ell+1/2}(\alpha R)} \frac{K_{\ell+1/2}(\alpha r)}{(\alpha r)^{1/2}} \quad \text{for} \quad r < R \quad ,$$
$$\phi_R^\ell(r) = \frac{(\alpha R)^{1/2}}{I_{\ell+1/2}(\alpha R)} \frac{I_{\ell+1/2}(\alpha r)}{(\alpha r)^{1/2}} \quad \text{for} \quad r > R \quad , \tag{15}$$

providing the following sufficient condition

$$-\frac{1}{4\pi}\int d\Omega \int_0^\infty |\phi_R^\ell|^2 V(r,\Omega) r^2 dr \geq \alpha R^2 \left\{\frac{(\ell+1)}{\alpha R} + \frac{K_{\ell-1/2}(\alpha R)}{K_{\ell+1/2}(\alpha R)}\right\} +$$
$$+ \alpha R^2 \left\{\frac{\ell}{\alpha R} + \frac{I_{\ell+3/2}(\alpha R)}{I_{\ell+1/2}(\alpha R)}\right\} . \tag{16}$$

In the limit $\alpha \to 0$, the above condition reduces to Calogero's "best" [7] sufficient condition

$$-\frac{1}{R}\int_0^R \left(\frac{r}{R}\right)^{2\ell} V(r) r^2 dr - \frac{1}{R}\int^\infty \left(\frac{R}{r}\right)^{2\ell+2} V(r)\, r^2\, dr \geq 2\ell+1 \quad . \tag{17}$$

Remark – The technical assumption on V needed for the validity of our arguments is $V \in L^2_{\ell oc}(\mathbb{R}^\nu)$ and $\lim_{r\to\infty} V(\vec{r}) = 0$. This ensures: a) Hunziker's theorem [1,2], i.e., $\varepsilon_2 = 0$ and b) since $L^2_{\ell oc}(\mathbb{R}^\nu) \subset L^1_{\ell oc}(\mathbb{R}^\nu)$ all integrals are well defined or equivalently, all trial wave functions ϕ_R^α are in the form

domain. Notice, also, that the condition of being "globally attractive" $\int V d^\nu x < 0$ can be generalised as follows: there exist $R > 0$ and $I > 0$ such that

$$\int_{|\vec{x}| \leq R'} V(\vec{x})\, d^\nu x \leq -I \quad \text{for all} \quad R' \geq R \quad , \tag{18}$$

thus avoiding the requirement of integrability.

3. BOUND-STATES IN THE N-BODY PROBLEM[†]

Sufficient conditions for the existence of bound-states for all $N \geq 2$ are derived inductively on N, that is, assuming that a certain sufficient condition for the existence of bound-states for $N = 2$ is verified, and assuming the existence of a bound-state of N particles, we prove the existence of a bound-state of N+1 particles.

3A. One- and Two-Dimensions

Let all two-body interactions V_{ij} be globally attractive, i.e. $\int V_{ij}\, d^\nu x < 0$. Consider now the (N+1)-body system. Denoting by $\vec{r}_i$ and m_i, $i = 1,\ldots,N+1$ the particles coordinates and masses, and introducing Jacobi coordinates [9]

$$\vec{\xi}_i = \vec{r}_{i+1} - \left(\sum_{j\leq i} m_j\right)^{-1} \left(\sum_{j\leq i} m_j\, \vec{r}_j\right) \quad , \quad i = 1,\ldots,N \quad ,$$

[†] From now on we shall use $\hbar=1$.

the Hamiltonian H_{N+1} reads

$$H_{N+1} = - \sum_{i=1}^{N} \frac{1}{2\mu_i} \Delta_{\xi_i} + \sum_{i<j} V_{ij}(\vec{r}_i - \vec{r}_j) \quad ,$$

where $\mu_i^{-1} = m_{i+1}^{-1} + \left(\sum_{j \leq i} m_j \right)^{-1}$,

that is

$$H_{N+1} = H_N - (2\mu_N)^{-1} \Delta_{\xi_N} + \sum_{i=1}^{N} V_{i,N+1} (\vec{r}_{N+1} - \vec{r}_i) \quad ,$$

where the Hamiltonian H_N involves only the coordinates $\vec{\xi}_1 , \ldots , \vec{\xi}_{N-1}$.

Let E_N be the energy of the bound-state of N particles (with $E_N < \varepsilon_N$) and $\Phi_N(\vec{\xi}_1 , \ldots , \vec{\xi}_{N-1})$ its normalised wave function. Consider then

$$\Phi(\vec{\xi}_1 , \ldots , \vec{\xi}_{N-1} , \vec{\xi}_N) = \Phi_N(\vec{\xi}_1 , \ldots , \vec{\xi}_{N-1}) \, \phi(\vec{\xi}_N) \quad ,$$

where ϕ is going to be conveniently chosen.

It is clear that

$$(\Phi , H_{N+1} \Phi) = E_N + (\phi , (H_0 + U_N) \phi) \quad ,$$

where $H_0 = - \frac{\Delta_{\xi_N}}{2\mu_N}$ and

$$U_N(\vec{\xi}_N) = \sum_{i=1}^{N} \int |\Phi_N(\vec{\xi}_1 , \ldots , \vec{\xi}_{N-1})|^2 \, V_{i,N+1}(\vec{r}_{N+1} - \vec{r}_i) \, d^\nu \xi_1 \ldots d^\nu \xi_{N-1}$$

is the "effective" potential seen by the (N+1)th particle in the presence of the bound-state of the other N particles. The important property of the effective potential U_N is that it is also globally attractive. This follows from

$$\int U_N(\vec{\xi}_N) \, d^\nu \xi_N = \sum_{i=1}^{N} \int d^\nu \xi_1 \ldots d^\nu \xi_{N-1} \, |\Phi_N(\vec{\xi}_1 , \ldots , \vec{\xi}_{N-1})|^2 \int d^\nu \xi_N V_{i,N+1}(\vec{\xi}_N + \vec{u}_i) \; ,$$

where $\vec{u}_i$ is a linear combination of $\vec{\xi}_1 , \ldots , \vec{\xi}_{N-1}$. The integral in ξ_N can be performed and is independent of $\vec{u}_i$, and since $\int |\Phi_N|^2 d^\nu \xi_1 \ldots d^\nu \xi_{N-1} = 1$ we obtain

$$\int U_N(\vec{\xi}_N) \, d^\nu \xi_N = \sum_{i=1}^{N} \left[\int V_{i,N+1}(\vec{x}) \, d^\nu \vec{x} \right]$$

which proves the statement. By a simple limiting argument, this result follows even if we use the more general definition of globally attractive potentials introduced in (18).

So, from the discussion in section 2 it then follows that ϕ can be chosen such that $(\phi , (H_0 + U_N) \phi) < 0$. For this choice of ϕ we have

$$(\Phi , H_{N+1} \Phi) < E_N < \varepsilon_N \quad .$$

Notice that the arguments can be repeated for any decomposition of the (N+1)-body system into 2 clusters of N and 1 particle(s) respectively. So, numbering the particles in such a way that E_N is the smallest energy of all N-body bound-states it then follows that

$$(\Phi , H_{N+1} \Phi) < \varepsilon_{N+1} \quad ,$$

concluding the proof.

It is easy to generalize the above result [3] to prove the existence of bound-states of N-particle systems for $\nu = 1$ and 2, provided there exists a decomposition of the system into two disjoint clusters

$$C_1 = \{i_1 , \ldots , i_{N_1}\} \quad , \quad C_2 = \{j_1 , \ldots , j_{N_2}\} \quad , \quad N_1 + N_2 = N \quad ,$$

both admitting bound-states with energies E^{C_1} and E^{C_2} (below the respective continuum thresholds) such that the "effective" intercluster potential

$$V^{eff}_{C_1C_2}(\vec{\xi}) = \sum_{\substack{i \in C_1 \\ j \in C_2}} V_{ij}(\vec{\xi}) \qquad (19)$$

is globally attractive and such that the continuum spectrum of H_N starts at $\varepsilon_N = E^{C_1} + E^{C_2}$.

In fact, the Hamiltonian H_N can be written as

$$H_N = H^{C_1} + H^{C_2} + \left[H_0 + \sum_{\substack{i \in C_1 \\ j \in C_2}} V_{ij}(\vec{r}_i - \vec{r}_j) \right]$$

where $H_0 = - \frac{\Delta_\xi}{2\mu}$, $\mu^{-1} = \left(\sum_{i \in C_1} m_i \right)^{-1} + \left(\sum_{j \in C_2} m_j \right)^{-1}$, and $\vec{\xi}$ denotes the position of the CM of C_2 with respect to the CM of C_1.

Consider now

$$\Phi\left(\vec{\xi}^{C_1}_1 , \ldots , \vec{\xi}^{C_1}_{N_1-1} , \vec{\xi}^{C_2}_1 , \ldots , \vec{\xi}^{C_2}_{N_2-1} , \vec{\xi}\right) =$$

$$= \Phi^{C_1}\left(\vec{\xi}^{C_1}_1 , \ldots , \vec{\xi}^{C_1}_{N_1-1}\right) \Phi^{C_2}\left(\vec{\xi}^{C_2}_1 , \ldots , \vec{\xi}^{C_2}_{N_2-1}\right) \phi(\vec{\xi}) \quad ,$$

where $\vec{\xi}^{C_\ell}_i$, $i = 1 , \ldots , N_{\ell-1}$ are the Jacobi coordinates for cluster C_ℓ , $\ell = 1, 2$, Φ^{C_ℓ} are the normalised eigenfunctions of H^{C_ℓ} (with energies E^{C_ℓ}) and, as before, ϕ will be conveniently chosen. For this trial function we have

$$(\Phi , H_N \Phi) = E^{C_1} + E^{C_2} + (\phi , (H_0 + V^{eff}_{C_1C_2}) \phi) \quad ,$$

where

$$V^{eff}_{C_1C_2}(\vec{\xi}) = \sum_{\substack{i \in C_1 \\ j \in C_2}} \int \left| \Phi^{C_1}\left(\vec{\xi}^{C_1}_1 , \ldots , \vec{\xi}^{C_1}_{N_1-1}\right) \right|^2 \left| \Phi^{C_2}\left(\vec{\xi}^{C_2}_1 , \ldots , \vec{\xi}^{C_2}_{N_2-1}\right) \right|^2 V_{ij}(\vec{r}_i - \vec{r}_j) \times$$

$$\times \prod_{k=1}^{N_1-1} d^\nu \xi^{C_1}_k \prod_{\ell=1}^{N_2-1} d^\nu \xi^{C_2}_\ell$$

is the "effective" intercluster potential (19). This follows from

$$\int V^{eff}_{C_1C_2}(\vec{\xi})d^\nu\xi = \sum_{\substack{i\in C_1\\ j\in C_2}} \int \prod_{k=1}^{N_1-1} d^\nu \xi_k^{C_1} \left|\phi^{C_1}\left(\vec{\xi}_1^{C_1},\ldots,\vec{\xi}^{C_1}_{N_1-1}\right)\right|^2 \times$$

$$\times \int \prod_{\ell=1}^{N_2-1} d^\nu \xi_\ell^{C_2}\left|\phi^{C_2}\left(\vec{\xi}_1^{C_2},\ldots,\vec{\xi}^{C_2}_{N_2-1}\right)\right|^2 \int d^\nu\xi\, V_{ij}\left(\vec{\xi}+\vec{u}_i^{C_1}+\vec{u}_j^{C_2}\right) ,$$

where $\vec{u}_i^{C_\ell}$, $\ell = 1, 2$, are linear combinations of $\vec{\xi}_1^{C_\ell},\ldots,\vec{\xi}^{C_\ell}_{N_\ell-1}$. The integral in ξ can be performed and is independent of $\vec{u}_i^{C_\ell}$, and as the functions ϕ_ℓ^C are normalised, we obtain

$$\int V^{eff}_{C_1C_2}(\vec{\xi})d^\nu\xi = \sum_{\substack{i\in C_1\\ j\in C_2}} \int V_{ij}(\vec{\xi})d^\nu\xi .$$

So, as the "effective" intercluster potential $V^{eff}_{C_1C_2}$ is assumed to be globally attractive, from the discussion in section 2 it follows that ϕ can be chosen such that $(\phi, (H_0 + V^{eff}_{C_1C_2})\phi) < 0$. For this choice of ϕ we have

$$(\phi, H_N\phi) < E^{C_1} + E^{C_2} = \varepsilon_N ,$$

thus concluding the proof.

Again, by a simple limiting argument this result follows even if we use the more general definition of globally attractive potentials introduced in (18).

3B. Three-Dimension

Let $C_1 = \{i_1,\ldots,i_{N_1}\}$ and $C_2 = \{j_1,\ldots,j_{N_2}\}$ be two disjoint clusters, $N_1+N_2 = N$. The following set of relative coordinates will prove suitable for displaying the binding mechanism that we exploit in $\nu = 3$:

$$\begin{aligned} \vec{x}_i &= \vec{r}_i - \vec{r}_{N_1} , \quad && i = i_1,\ldots,i_{N_1-1} , \\ \vec{y}_j &= \vec{r}_{N_1+j} - \vec{r}_N , && j = j_1,\ldots,j_{N_2-1} , \\ \vec{z} &= \vec{r}_{N_1} - \vec{r}_N , && \end{aligned} \tag{20}$$

i.e., inside each cluster C_ℓ , $\ell = 1, 2$, we single out a given particle (for simplicity we always make it the last one in the cluster) and take coordinates relative to these particles.

In the appendix we show that the Hamiltonian H_N written in terms of (20) is given by

$$H_N = H^{C_1} + H^{C_2} + H^D + \tilde{V}_{C_1C_2} + T_{H-E} , \tag{21}$$

where D is the two-body cluster:

$$D = \{i_{N_1}, j_{N_2}\} \equiv \{N_1, N\} ,$$

$$\tilde{V}_{C_1C_2} = \sum_{\substack{k=1,\ldots,N_1-1\\ \ell=1,\ldots,N_2-1}} V_{i_k j_\ell}(\vec{x}_{i_k} - \vec{y}_{j_\ell} + \vec{z}) \tag{22}$$

is the intercluster interaction excluding V_{N_1N} already included in H^D, and

$$T_{H-E} = \sum_{\ell=1}^{N_1-1} \frac{\vec{p}.\vec{k}_{i_\ell}}{2m_{N_1}} - \sum_{\ell=1}^{N_2-1} \frac{\vec{p}.\vec{q}_{j_\ell}}{2m_N} \qquad (23)$$

is the Hughes-Eckart kinetic energy [1]. Here $\vec{k}_i$, $\vec{q}_j$ and $\vec{p}$ denote the canonically conjugate momenta to $\vec{x}_i$, $\vec{y}_j$ and $\vec{z}$ respectively.

Let us now assume that there are eigenstates Φ^{C_i} of H^{C_i}, with energies $E^{C_i} < \varepsilon^{C_i}$, $i = 1, 2$, where ε^C denotes the continuum threshold of H^C.

Considering then a state Φ_N of the N-body system of the form:

$$\Phi_N = \Phi^{C_1}(\vec{x}_1, \dots, \vec{x}_{N_1-1})\, \Phi^{C_2}(\vec{y}_1, \dots, \vec{y}_{N_2-1})\, \phi(\vec{z}) \quad , \qquad (24)$$

we get

$$(\Phi_N, H_N \Phi_N) = E^{C_1} + E^{C_2} + (\phi, H^D \phi) + (\Phi_N, \tilde{V}_{C_1C_2} \Phi_N) + (\Phi_N, T_{H-E} \Phi_N) \quad . \qquad (25)$$

Now, for purely attractive potentials V_{ij}

$$(\Phi_N, \tilde{V}_{C_1C_2} \Phi_N) \leq 0 \quad , \qquad (26)$$

and, if it is possible to choose ϕ such that

$$(\phi, H^D \phi) \leq 0 \qquad (27)$$

and

$$(\phi, \vec{p}\, \phi) = 0 \quad , \qquad (28)$$

then

$$(\Phi_N, H_N \Phi_N) < E^{C_1} + E^{C_2} \quad . \qquad (29)$$

Comments -

(a) Condition (27) is a sufficient condition for the existence of a bound-state for the cluster $D \equiv \{N_1, N\}$.

(b) Condition (28) follows from symmetry requirements on ϕ.

(c) Both (27) and (28) are automatically satisfied if the V_{ij}'s are central potentials and ϕ is taken to be a bound-state of H^D with well defined angular momentum.

(d) For non-central potentials a sufficient condition for the possibility of choosing ϕ satisfying (27) and (28) is for instance (11).

Under the above assumptions on V_{ij} we can draw the following conclusions [8]:

(i) The threshold for the continuum spectrum is given by a two-cluster break up:

$$\varepsilon_N = \inf_{\substack{C_1 \cup C_2 = \{1,\dots,N\} \\ C_1 \cap C_2 = \phi}} (E_0^{C_1} + E_0^{C_2}) \quad , \tag{30}$$

where E_0^C denotes the ground state energy (the infimum of the spectrum) of H^C.

(ii) For all $N \geq 2$ there exists an eigenstate of H_N with energy $E_N < \varepsilon_N$.

Proof -

(i) Suppose the minimum in (2) were attained at $\ell \geq 3$, i.e.

$$\varepsilon_N = E_0^{C_1} + E_0^{C_2} + \dots + E_0^{C_\ell} \quad .$$

Now, from (29) it follows that $E_0^{C_1 \cup C_2} < E_0^{C_1} + E_0^{C_2}$, therefore the only possibility left is $\ell = 2$.

(ii) Follows trivially from (1) and (29).

4. IDENTICAL PARTICLES - LARGE N RESULTS

The results of the previous section have one thing in common: they all assume the existence of bound-states of two-particles. However, even if the two-body interaction is not strong enough for binding two-particles, there is a class of interactions for which we can prove the existence of bound-

states of N particles, provided N is large enough. To prove this let us consider a system of N identical bosons of mass m interacting via a two-body potential V. Using relative coordinates with respect to the N^{th} particle, $\vec{x}_i = \vec{r}_i - \vec{r}_N$, $i = 1,\dots,N-1$, the Hamiltonian H_N reads:

$$H_N = \sum_{i=1}^{N-1} \frac{\vec{k}_i^2}{m} + \sum_{i=1}^{N-1} V(\vec{x}_i) + \frac{1}{2} \sum_{i \neq j=1}^{N-1} V(\vec{x}_i - \vec{x}_j) + + \frac{1}{2} \sum_{i \neq j=1}^{N-1} \frac{\vec{k}_i . \vec{k}_j}{2m} \quad . \tag{31}$$

For the trial wave function

$$\psi_N(\vec{x}_1, \dots, \vec{x}_{N-1}) = \prod_{i=1}^{N-1} \phi(\vec{x}_i) \tag{32}$$

we have

$$(\psi_N , H_N \psi_N) = (N-1) \left\{ (\phi , H_2 \phi) + \frac{(N-2)}{2} u(\phi) \right\} \tag{33}$$

where $$u(\phi) = \int |\phi(\vec{x}_1)\phi(\vec{x}_2)|^2 \, V(\vec{x}_1 - \vec{x}_2) \, d^3x_1 \, d^3x_2 \quad . \tag{34}$$

Notice that the Hughes-Eckart terms disappear by symmetry.

It is clear from (33) that if there is a $\phi \in L^2(\mathbb{R}^\nu)$ such that $u(\phi) < 0$ then for $N > N_0(\phi) = 2 \left\{ 1 + \frac{(\phi, H_2\phi)}{|u(\phi)|} \right\}$, it follows that $(\psi_N, H_N \psi_N) < 0$. So, for some $N_0 \leq \inf_{||\phi||} N_0(\phi)$ (by Hunziker's theorem) H_{N_0} will have a bound-state. Sufficient conditions on V for this to happen are:

(a) V is purely attractive ($V(\vec{x}) \leq 0$): in this case $u(\phi) \leq 0$ for all ϕ.

(b) V has an attractive core, i.e. $V(\vec{x}) \leq 0$ for $|\vec{x}| \leq R$: choosing $\phi(\vec{x}) = 0$ for $|\vec{x}| \geq R/2$ we have $u(\phi) \leq 0$.

(c) V is globally attractive: let $\nu = 3$ (the case $\nu = 1$ or 2 has been discussed in section 3A). For $\phi_\beta(\vec{x}) = \beta^{\nu/2} \phi(\beta\vec{x})$ we have

$$u(\phi_\beta) = \beta^{2\nu} \int V(\vec{x}-\vec{y}) |\phi(\beta\vec{x})|^2 \, |\phi(\beta\vec{y})|^2 d^\nu x \, d^\nu y =$$

$$= \beta^\nu \int V(\vec{x} - \frac{\vec{y}}{\beta}) \, |\phi(\beta\vec{x})|^2 \, |\phi(\vec{y})|^2 d^\nu x \, d^\nu y =$$

$$= \beta^\nu \int d^\nu y |\phi(\vec{y})|^2 \left[\int V(\vec{\omega}) \, |\phi(\beta\vec{\omega}+\vec{y})|^2 d^\nu \omega\right] ,$$

where $\vec{\omega} = \vec{x}-\vec{y}/\beta$.

Since

$$\lim_{\beta\to 0} \left[\int V(\vec{\omega}) \, |\phi(\beta\vec{\omega}+\vec{y})|^2 d^\nu \omega\right] = |\phi(\vec{y})|^2 \int V(\vec{\omega}) \, d^\nu \omega ,$$

if $\int V(\vec{\omega}) \, d^\nu \omega < 0$,

$u(\phi_\beta) < 0$ for β small enough.

We now show the existence of an infinite sequence $2 \leq N_0 \leq N_1 \ldots \leq N_n \ldots$ such that H_{N_i} has at least one bound-state. Suppose this sequence is finite, i.e. $\exists$ L such that for $N > N_L$, H_N has no bound-state. In fact, by Hunziker's theorem [1,2], the continuum threshold is given by

$$\varepsilon_N = \sum_{i=1}^{L} k_i E_{N_i} , \qquad k_i \text{ integer}$$

where

$$H_{N_i} \psi_{N_i} = E_{N_i} \psi_{N_i} .$$

Now, since

$$\sum_{i=1}^{L} k_i N_i \leq N$$

it follows that

$$k_i \leq N , \qquad i = 1,\ldots,L ,$$

so that

$$\varepsilon_N > N \sum_{i=1}^{L} E_{N_i} . \tag{35}$$

As we have seen (33) in the case of bosons $(\psi_N, H_N \psi_N)$ has $-CN^2$ $(C>0)$ for upperbound. Due to the linear dependence of ε_N on N it follows that there exist a $N > N_L$ such that H_N

has at least one bound-state, thus proving that the sequence of N_i such that H_{N_i} has at least one bound-state is infinite.

In fact all these interactions are catastrophic or collapsing since the binding energy per particle diverges as $N \to \infty$. Any two-body interaction is catastrophic if it is not <u>stable</u> [5] (a two-body interaction V is said to be stable if there exists a constant $B \geq 0$ such that $U(\vec{r}_1, \ldots, \vec{r}_N) = \frac{1}{2} \sum_{i \neq j=1}^{N} V(\vec{r}_i - \vec{r}_j) \geq -BN$ for all $N \geq 2$ and for all $\vec{r}_1, \ldots, \vec{r}_N \in \mathbb{R}^\nu$). As proved in page 35 of Ruelle's book [5], if the interaction is not stable there exist a sequence of integers $N_1 < N_2 < \ldots < N_k < N_{k+1} \ldots$, a sequence of points $\vec{\xi}_1, \ldots, \vec{\xi}_k \in \mathbb{R}^\nu$ and constants $C > 0$, $\delta > 0$ such that

$$U(\vec{r}_1 = 0, \ldots, \vec{r}_{N_k}) \leq -C N_k^2 \quad \text{if} \quad |\vec{r}_i - \vec{\xi}_i| < \delta \quad \text{for all} \quad i.$$

Then, it is easy to construct a sequence of normalised wave functions such that

$$(\psi_{N_k}, V \psi_{N_k}) < -C N_k^2 \quad \text{and}$$

$$(\psi_{N_k}, H_0 \psi_{N_k}) \leq \begin{cases} C' N_k & \text{(bosons)} \\ C'' N_k^{(1+\frac{2}{\nu})} & \text{(fermions)} \end{cases} .$$

Therefore there exists a $k_0 \geq 1$ such that $(\psi, H_{N_{k_0}} \psi) < 0$

and so there exists $2 \leq N_0 \leq N_{k_0}$ such that for fermions H_{N_0} has at least a bound-state if $\nu \geq 3$ (and for bosons if $\nu \geq 1$, as already seen).

APPENDIX - KINEMATICS FOR $\nu = 3$

Here we derive expression (21) of H_N in terms of the relative coordinates $\vec{x}_i$, $\vec{y}_i$ and $\vec{z}$ defined on (20). The only thing we have to do to obtain expression (21) is to express the kinetic energy in terms of the momenta $\vec{k}_i$, $\vec{q}_j$ and $\vec{p}$ canonically conjugate to $\vec{x}_i$, $\vec{y}_j$ and $\vec{z}$ respectively. This is easily done since the transformation from the momenta $\vec{k}_i$, $\vec{q}_j$ and $\vec{p}$ to the momenta $\vec{p}_{r_i}$ canonically conjugate to $\vec{r}_i$ is given by

$$\begin{bmatrix} \vec{p}_{r_1} \\ \vdots \\ \vec{p}_{r_{N_1}} \\ \vdots \\ \vec{p}_{r_N} \end{bmatrix} = \begin{bmatrix} \tilde{B} \end{bmatrix} \begin{bmatrix} \vec{P}_{CM} = 0 \\ \vec{k}_{i_1} \\ \vdots \\ \vec{k}_{i_{N_1-1}} \\ \vdots \\ \vec{q}_{j_i} \\ \vdots \\ \vec{q}_{j_{N_2-1}} \\ \vec{p} \end{bmatrix} , \qquad \text{(A1)}$$

where $\vec{P}_{CM} = 0$ is the CM momentum canonically conjugate to $\vec{R}_{CM} = 0 = \sum_{i=1}^{N} \frac{m_i \vec{r}_i}{M}$ $(M = \sum_{i=1}^{N} m_i)$ and the matrix $\tilde{B}$ is the

transpose of the matrix B defining transformation (20):

$$\begin{bmatrix} \vec{R}_{CM} = 0 \\ \vec{x}_{i_1} \\ \vdots \\ \vec{x}_{i_{N_1-1}} \\ \vec{y}_{j_1} \\ \vdots \\ \vec{y}_{j_{N_2-1}} \\ \vec{z} \end{bmatrix} = \begin{bmatrix} \frac{m_1}{M} & \frac{m_2}{M} & \cdots & \frac{m_{N_1}}{M} & \cdots & \frac{m_N}{M} \\ 1 & 0 & \cdots\ 0 & -1 \quad 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & -1 \quad 0 & \cdots & 0 \\ 0 & \cdots & 0 \quad 0 & 1 \quad 0 & \cdots & 0 \quad -1 \\ \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 0 \quad 0 & 0 & \cdots & 1 \quad -1 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \quad -1 \end{bmatrix} \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_{N_1} \\ \vdots \\ \vec{r}_N \end{bmatrix} .$$

So, using (A1), after a straightforward calculation we obtain

$$H_N = H^{C_1} + H^{C_2} + H^D + \tilde{V}_{C_1C_2} + T_{H-E} ,$$

where

$$H^{C_1} = \sum_{\ell=1}^{N_1-1} \frac{\vec{k}_{i_\ell}^2}{2\mu_{N_1 i_\ell}} + \sum_{k>\ell=1}^{N_1-1} \frac{\vec{k}_{i_k} \cdot \vec{k}_{i_\ell}}{m_{N_1}} + \sum_{\ell=1}^{N_1-1} V_{i_\ell N_1}(\vec{x}_{i_\ell}) +$$

$$+ \sum_{k>\ell=1}^{N_1-1} V_{i_k i_\ell}(\vec{x}_{i_k} - \vec{x}_{i_\ell}) ,$$

$$H^{C_2} = \sum_{\ell=1}^{N_2-1} \frac{\vec{q}^{\,2}_{j_\ell}}{2\mu_{Nj_\ell}} + \sum_{k>\ell=1}^{N_2-1} \frac{\vec{q}_{j_k}\cdot\vec{q}_{j_\ell}}{m_N} + \sum_{\ell=1}^{N_2-1} V_{j_\ell N}(\vec{y}_{j_\ell}) + \sum_{k>\ell=1}^{N_2-1} V_{j_k j_\ell}(\vec{y}_{j_k}-\vec{y}_{j_\ell}) \quad ,$$

$$H^D = \frac{\vec{p}^{\,2}}{2\mu_{N_1N}} + V(\vec{z}) \quad ,$$

$$T_{H-E} = \sum_{\ell=1}^{N_1-1} \frac{\vec{p}\cdot\vec{k}_{i_\ell}}{2m_{N_1}} - \sum_{\ell=1}^{N_2-1} \frac{\vec{p}\cdot\vec{q}_{j_\ell}}{2m_N} \quad ,$$

$$\tilde{V}_{C_1C_2} = \sum_{\substack{k=1,\dots,N_1-1\\ \ell=1,\dots,N_2-1}} V_{i_k j_\ell}(\vec{x}_{i_k} - \vec{y}_{j_\ell} + \vec{z}) \quad ,$$

μ_{ij} being the reduced masses.

REFERENCES

1. M. Reed and B. Simon, Methods of Modern Mathematical Physics, vol. IV, Academic Press (1978) N.Y.
2. W. Hunziker, Helv. Phys. Acta 39, 451 (1966).
3. F.A.B. Coutinho, C.P. Malta and J. Fernando Perez, Phys. Lett. 97A, 242 (1983).
4. B. Simon, Ann. of Phys. 97, 279 (1976). M. Klaus, Ann. of Phys. 108, 288 (1977).
5. D. Ruelle, Statistical Mechanics, Mathematical Physics Monograph Series, W.A. Benjamin, Inc. (1969), Massachussets.
6. K. Chadan and A. Martin, J. Physique - Lettres 41, L-205 (1980).
7. F. Calogero, J. Math. Phys. 6, 161 (1965).
8. F.A.B. Coutinho, C.P. Malta and J. Fernando Perez, Phys. Letters 100A, 460 (1984).
9. M. Reed and B. Simon, Methods of Modern Mathematical Physics, vol. III, Academic Press (1979) N.Y.