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## ELECTRON-POSITRON ANNIHILATION AND NON-ABELIAN EIKONAL EXPONENTIATION

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ABSTRACT

Using the non-abelian exponentiation theorem, we have calculated to fourth order in perturbative QCD the exponent which contains the contributions of the annihilation process into a pair quark-antiquark, including soft gluon production up to a maximum energy $\Delta$. We suggest a generalization of these results to all orders, which contains all leading as well as non-leading logarithms of $\frac{m}{\sqrt{S}}$, where $m$ denotes the quark mass and $\sqrt{S}$ represents the center of mass energy of the electron-positron pair. We show in particular that the dependence of the exponent on these parameters is governed, in leading order, by the effective coupling $\bar{g}\left(\frac{\Delta m}{\mu\sqrt{S}}\right)$ as given by the renormalization group.



I. INTRODUCTION

The problem of calculating higher order contributions to high energy reactions in perturbative QCD has been investigated by many authors [1]. In various cases it was found that the corrections can become large and it has been suggested that these may be summed to all orders.

In this paper we study the electron-positron annihilation process and consider a procedure for summing higher order corrections via the non-abelian eikonal exponentiation. As is well known, in QED soft photon amplitudes and cross sections are completely given in terms of exponentials of order $e^2$ quantities [2]. The derivation of these results uses the eikonal approximation and the eikonal identity. In QCD, much progress has been made towards a proof on the exponentiation of the leading [3] as well as the non-leading logs [4] and recently [5] complete results have been proved for all logarithms.

The theorem concerning the non-abelian eikonal exponentiation is as follows [5]. Let $X$ be a gauge invariant, physical quantity, related to a cross section evaluated in the eikonal approximation. Then

$$X = \exp(Y) \tag{1}$$

where $Y$ is calculated in perturbation theory from an infinite series of terms, each of which corresponds to a single Feynman diagram. The diagrams contributing to $Y$ are a subset of those contributing to $X$, with colour weights which are in general different from those of the corresponding terms in $X$. For instance, to fourth order, the colour weights in $Y$

correspond to a maximally non-abelian colour given by $C_F C_A$. These Casimir operators are defined by $t_a t_a = C_F I$; $f_{abc} f_{abd} = C_A \delta_{cd}$, where $t_a$ are the representation matrices for the fermions and $f_{abc}$ are the structure constants.

When soft gluon production up to a maximum energy $\Delta$ is included:

$$\sum_i K_i < \Delta \tag{2a}$$

the quantity $X$ is closely connected with a cross section like:

$$e^+ e^- \longrightarrow \gamma^* \longrightarrow q\,\bar{q} + \text{soft gluons} \tag{2b}$$

The condition (2a) can be enforced with the help of the step function $\theta[\Delta - \sum_i K_i]$. Using the integral representation of the $\theta$-function, we write the physical cross section as follows:

$$\sigma_{ph}\left(\frac{m}{\sqrt{S}}, \frac{\Delta}{\mu}\right) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dy}{y - i\epsilon}\, e^{i\Delta y}\, \bar{\sigma}\left(\frac{m}{\sqrt{S}}, y\,\mu\right) \tag{3a}$$

Here $m$ is the quark mass, $\sqrt{S}$ is the invariant center of mass energy, and $\mu$ is a unit of mass which appears in connection with the U.V. renormalization substraction. Furthermore, $\bar{\sigma}$ is also an IR finite quantity because the Bloch-Nordsieck theorem [6] applies to process (2). The exponentiation theorem applies to $\bar{\sigma}$. We have [5]:

$$\bar{\sigma} = \left| f\left(\frac{m}{\sqrt{S}}\right)\right|^2 \exp\left[ Y\left(\frac{m}{\sqrt{S}}, y\mu, \bar{g}(\mu)\right)\right] \tag{3b}$$

where $f$ is a renormalization vertex-function.



In this way one obtains a differential equation satisfied by $\bar{\sigma}$:

$$y \frac{d}{dy} \bar{\sigma} = \bar{G}\left[\frac{m}{\sqrt{S}}, \bar{g}(y\mu)\right] \bar{\sigma} \tag{4a}$$

where:

$$\bar{G} = y \frac{d}{dy} Y \tag{4b}$$

The particular form of the y-dependence of $\bar{G}$, via the effective coupling constant $\bar{g}$, is just a consequence of the renormalizability of the eikonal approximation. Equations like (4) have been conjectured previously also in other investigations [7].

Since $\bar{G}$ controls the behaviour of the cross section, it is important to evaluate it, especially at high energy, where we expect perturbation theory to be applicable. The purpose of this work is to calculate the factor $\bar{G}$ up to fourth order terms, and try to guess a possible generalization valid to all orders in perturbative QCD. We obtained:

$$\bar{G}\left[\frac{m}{\sqrt{S}}, \bar{g}(\mu y)\right] = -\left\{ g^2\left[1 - \frac{11 g^2}{24\pi^2} C_A \ln\left(\frac{1}{i\mu y}\frac{m}{\sqrt{S}}\right)\right]\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\right] + g^4 C_A \bar{H}\left(\frac{m}{\sqrt{S}}\right)\right\} \frac{C_F \operatorname{Tr}(I)}{(2\pi)^2} \tag{5a}$$

Here $\beta$ is the speed of one quark in the rest frame of the other, and is related to $m/\sqrt{S}$ by the expression:

$$1 - \beta^2 = \left(\frac{s}{2m^2} - 1\right)^{-2} \tag{5b}$$

The function of $\beta$ in the second square bracket represents the bremsstrahlung probability function $B(\beta) \equiv B\left(\frac{m}{\sqrt{S}}\right)$. $\bar{H}\left(\frac{m}{\sqrt{S}}\right)$ has in general a complicated dependence on $\frac{m}{\sqrt{S}}$, through the dilogarithmic and three-logarithmic functions [8]. We will present the complete result later on, but here we only notice that at high energies the corresponding expression simplifies considerably and consequently $\bar{H}\left(\frac{m}{\sqrt{S}}\right)$ has the simple form:

$$\bar{H}\left(\frac{m}{\sqrt{S}}\right) = \bar{C}_0 + \bar{C}_1 \ln\left(\frac{m}{\sqrt{S}}\right) \tag{5c}$$

with $\bar{C}_0$ and $\bar{C}_1$ constants.

It is worthwhile to remark that the expression multiplying the bremsstrahlung function represents the beginning of the effective coupling $\bar{g}\left(\frac{m}{\sqrt{S}}\ \frac{1}{\mu y}\right)$. Although as noticed above, the dependence on $\mu y$ was to be expected, its dependence on $\frac{m}{\sqrt{S}}$ represents a result which was not antecipated.

In section II, we state the results of our calculation and point out some relevant features of the method employed which made possible to obtain them in a closed form. A few more details are presented in the appendixes. In section III we suggest a conjecture concerning the generalization of these results, and present an argument which supports the appearence of the effective coupling $\bar{g}^2\left(\frac{m}{\sqrt{S}}\ \frac{\Delta}{\mu}\right)$ to all orders in perturbative QCD. Finally, using the expression of the exponent evaluated to fourth order, we derive in section IV an explicit result for the cross section $\sigma_{ph}$ which contains all leading as well as non-leading logarithms.



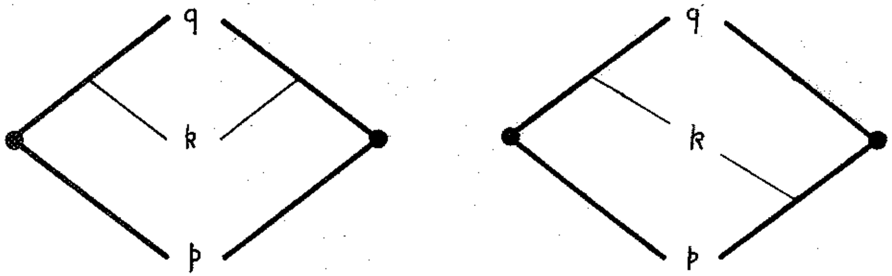
## II. RESULTS OF THE CALCULATIONS

In order to determine the factor $\bar{G}(\beta, y\mu)$ defined in equation (4) we will calculate in this section a closely related quantity, $G(\beta, \frac{\Delta}{\mu})$ defined by the equations:

$$\sigma_{ph} \equiv \theta(\Delta)\, \sigma\left(\beta, \frac{\Delta}{\mu}\right) \tag{6a}$$

$$\Delta \frac{d}{d\Delta} \sigma\left(\beta, \frac{\Delta}{\mu}\right) \equiv G\left(\beta, \frac{\Delta}{\mu}\right) \tag{6b}$$

To this end we work in the Feynman gauge and use consistently dimensional regularization in a space-time of dimension $d=4+\eta$. Since purely virtual diagrams are independent of $\Delta$, only graphs which have at least one real gluon, satisfying the condition (2a), will contribute to $G$. To set up the notation, consider the contribution of second order graphs shown in Figure 1.




(a) (b)

Fig.1.

The right hand side of each graph represents a contribution to the complex conjugate part of the amplitude, and a sum over symmetric diagrams with respect to reflections about vertical and horizontal axes are always to be understood. Thick lines are quarks, while thin lines denote gluons. The black blob represents the point where the virtual photon in equation (2) produces a pair quark-antiquark with momenta q and p, respectively. We denote the four-momenta of the gluon by k and its modulus $|\vec{k}|$ by K. Working in the rest frame of p, we represent by x the cosine of the angle between $\vec{k}$ and $\vec{q}$. In this notation, we obtain from these graphs the contribution:

$$\sigma^{(2)} = \frac{g^2 C_F \, \mathrm{Tr}(I)}{(2\pi)^{3+\eta}} \int_0^{\Delta} dk\, K^{\eta-1} \int d\Omega^{3+\eta} \left[ \frac{1}{1-\beta x} - 1 \right]$$

$$= \frac{g^2 C_F \, \mathrm{Tr}(I)}{(2\pi)^{3+\eta}} \left( \frac{\Delta}{\mu} \right)^{\eta} \frac{1}{\eta} \int d\Omega^{3+\eta} \frac{\beta x}{1-\beta x} \qquad (7a)$$

When applying $\Delta \frac{d}{d\Delta}$ in equation (6) in order to obtain $G^{(2)}$, we note that this operation produces in (7a) an extra factor of $\eta$ in the numerator. Hence we may put $\eta=0$ in all other places in the integral obtaining:

$$G^{(2)}(\beta) = \frac{g^2 C_F \, \mathrm{Tr}(I)}{(2\pi)^2} \left[ \frac{1}{\beta} \ln\left( \frac{1+\beta}{1-\beta} \right) - 2 \right] \qquad (7b)$$

In this simple case, $G^{(2)}$ is proportional to the bremsstrahlung function, being independent of $\Delta$. Furthermore, as remarked before, the factor G must be an infrared finite quantity.

We now turn to the evaluation of $G^{(4)}$ which results from contributions of fourth-order diagrams with colour



factors $C_F C_A$. Typical abelian type of diagrams, contributing to this order are shown in Figure 2.

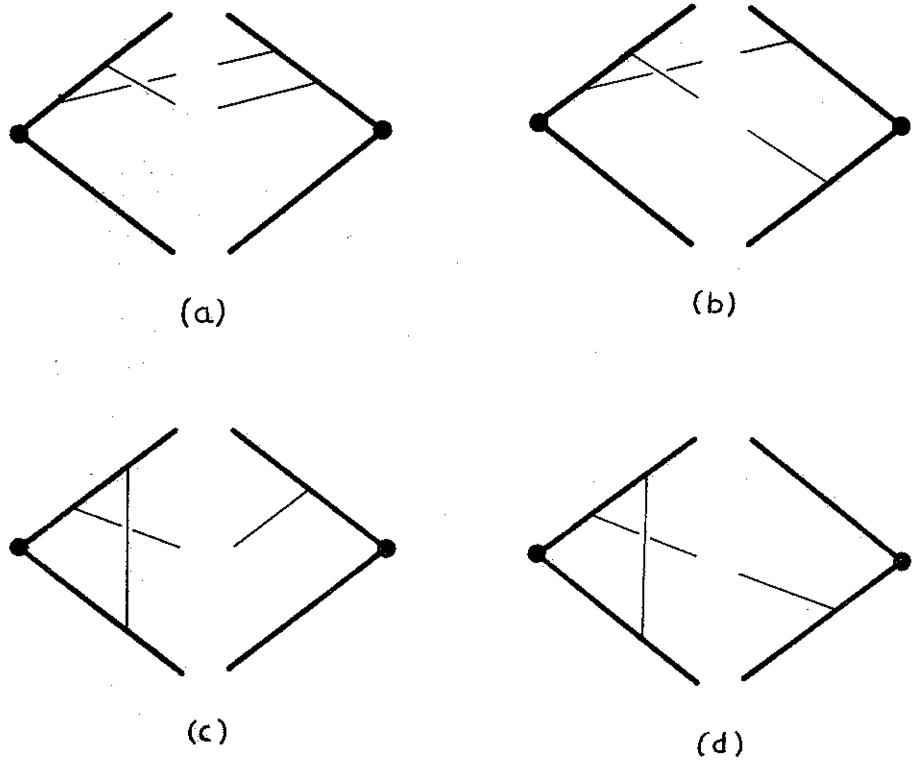



Fig. 2.

After a long calculation we obtain from these graphs the following contribution:

$$G^{(4)}_{ab}(\beta) = \frac{C_F C_A \, \mathrm{Tr}(I)}{(2\pi)^4} g^4 \left\{ -\frac{\pi^2}{2\beta} \left[ \frac{1}{\beta} \ln\left( \frac{1+\beta}{1-\beta} \right) - 2 \right] + \right.$$

$$+\frac{1}{2\beta^2}\ln^2\left(\frac{1+\beta}{1-\beta}\right)+\frac{1}{\beta^2}\left[Li_2\left(\frac{-2\beta}{1-\beta}\right)+Li_2\left(\frac{2\beta}{1+\beta}\right)\right]-$$

$$-\frac{1}{\beta^2}Li_3(1)+\frac{1}{2\beta^2}\left[Li_3\left(\frac{1+\beta}{1-\beta}\right)+Li_3\left(\frac{1-\beta}{1+\beta}\right)\right]-$$

$$-\frac{1}{4\beta^2}\ln\left(\frac{1+\beta}{1-\beta}\right)\left[Li_2\left(\frac{1+\beta}{1-\beta}\right)-Li_2\left(\frac{1-\beta}{1+\beta}\right)\right]\Bigg\} \qquad (8)$$

As expected, this result is infrared finite. Moreover, it is much more complicated than the corresponding second order contribution, due to the appearence of the dilogarithmic functions $Li_2$ and of the threelogarithmic functions $Li_3$. These functions and their properties are briefly discussed in appendix A. In this way, one can see that at high energies, when $\beta \to 1$, the terms in the last two lines of equation (8) will yield contributions proportional to $\ln^3(1-\beta)$.

Next we consider the contribution of topologically related nonabelian diagrams shown in Figure 3.

These diagrams are more difficult to evaluate, because they contain individually superleading infrared divergences of order $\eta^{-3}$ [10]. The main idea is to group these graphs in such a way as to cancel all singularities before actually starting the calculation. Then we can put $\eta=0$, a step which is essential for obtaining the contributions of these diagrams in closed form.

We will now outline the way in which the cancellation of the infrared singularities does actually occur. Using the notation already introduced, we obtain by a straightforward



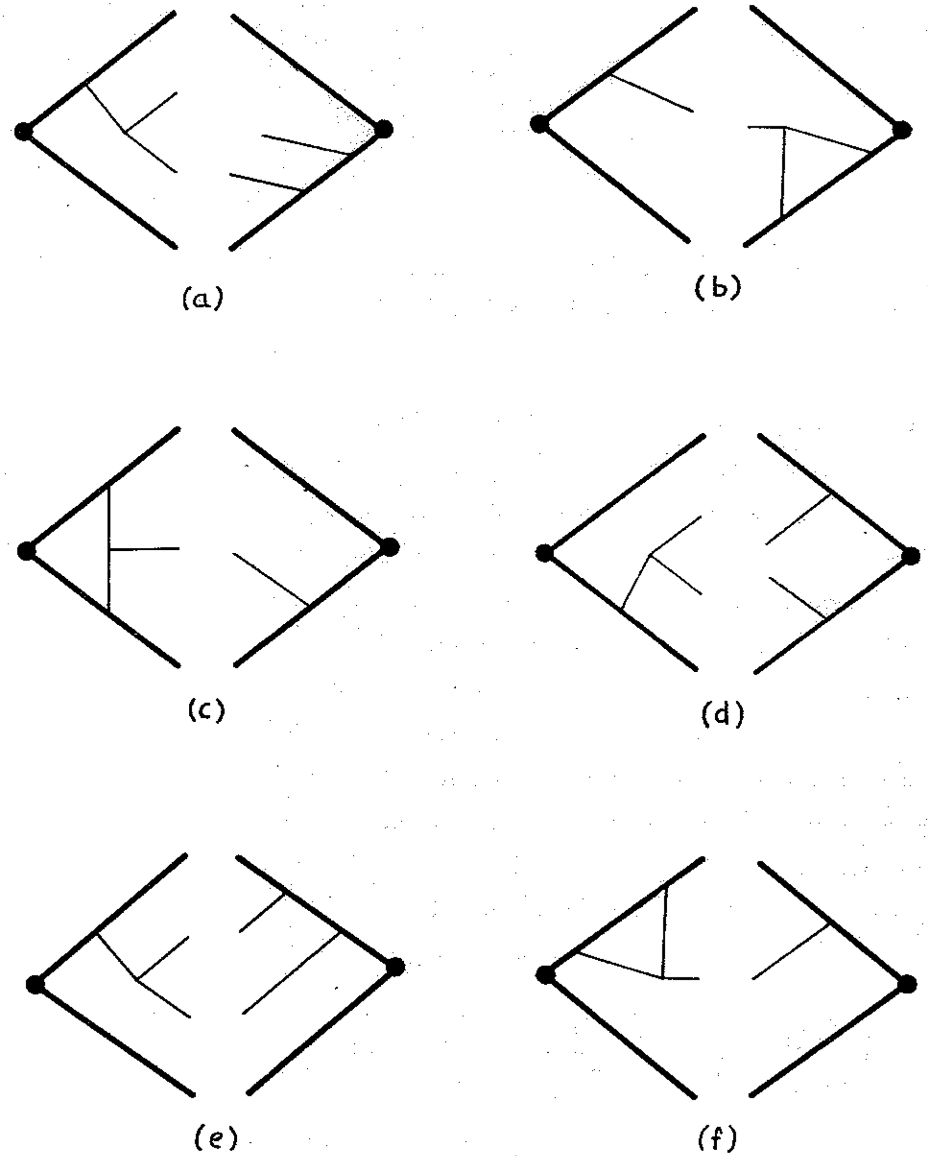



Fig. 3.

application of the Feynman rules that the contribution to the cross section of diagram (a) is given by

$$\sigma^{(a)} = \frac{g^4 C_F C_A \, Tr(I)}{16\,(2\pi)^{6+2\eta}} \int \frac{d^{3+\eta}k}{K^3} \int \frac{d^{3+\eta}k'}{(K')^2} \cdot$$

$$\cdot \left[ \frac{1}{K+K'} \; \frac{\beta(K'x'-Kx)}{K+K'-\beta(Kx+K'x')} \; \frac{1}{1-y} \right] \qquad (9a)$$

Here $k$ and $k'$ denote the momenta of the two gluons and $y = \hat{k}.\hat{k}'$. Furthermore in view of the condition (2a) we have $K + K' < \Delta$. The above expression exhibits a triple pole singularity: a factor $\eta^{-2}$ results from the region where both gluon momenta become soft with $K \to 0$, $K' \to 0$, and another factor $\eta^{-1}$ is associated with a parallel configuration of the gluons where $y \to 1$. In order to see the cancellation mentioned above, it will be convenient to separate equation (9a) in two parts: one which has a pole in the limit $y \to 1$ and another which is convergent in this limit. In this way we find the expression:

$$\sigma^{(a)} = \frac{g^4 C_F C_A \, Tr(I)}{16\,(2\pi)^{6+2\eta}} \int \frac{d^{3+\eta}k}{K^3} \int \frac{d^{3+\eta}k'}{(K')^2} \; \frac{(K'-K)}{(K'+K)^2} \; \frac{\beta x'}{1-\beta x'} \; \frac{1}{1-y} +$$

$$+ \frac{g^4 C_F C_A \, Tr(I)}{16\,(2\pi)^{6+2\eta}} \int \frac{d^{3+\eta}k}{K^2} \int \frac{d^{3+\eta}k'}{(K')^2} \; \frac{1}{(K+K')^2} \cdot$$

$$\cdot \left[ \frac{K-K'+2K'(1-\beta x')}{K(1-\beta x)+K'(1-\beta x')} \; \frac{1}{1-\beta x'} \; \frac{\beta(x'-x)}{1-y} \right] \qquad (9b)$$



The first expression in the above equation yields the triple pole singularity. On the other hand, the second expression gives only a single pole singularity, since both momenta $k$ and $k'$ are controlling.

Next consider the contribution of graph (b). Here the momenta of the real gluon satisfies the condition $K < \Delta$, but the momenta of the virtual gluon $k'$ is not limited. Performing the $k_0'$ integration using the Cauchy theorem we find:

$$\sigma^{(b)} = \frac{g^4 C_F C_A \, Tr(I)}{16\,(2\pi)^{6+2\eta}} \int \frac{d^{3+\eta}k}{K^3} \int \frac{d^{3+\eta}k'}{(K')^3} \cdot$$

$$\cdot \left[ \frac{\beta[Kx+2K'(x'-x)]}{(K'-K)(1-\beta x)} \; \frac{1}{1-y} \right] \qquad (10a)$$

We note that the first term in the numerator of (10), proportional to $Kx$, yields for the same reason as previously discussed a cubic singularity. For the contribution coming from this term, we split the range of integration in two parts: one where $(K+K') > \Delta$ which yields only a single singularity, and another where $K+K' < \Delta$ which gives the whole cubic singularity. Adding this last contribution with the first expression in equation (9b) we find:

$$- \frac{g^4 C_F C_A \, Tr(I)}{8\,(2\pi)^{6+2\eta}} \int \frac{d^{3+\eta}k}{K^2} \int \frac{d^{3+\eta}k'}{(K')^2} \cdot$$

$$\cdot \left[ \frac{1}{(K+K')^2} \; \frac{\beta x}{(1-\beta x)} \; \frac{1}{1-y} \right] \qquad (10b)$$

Notice that the superleading singularity cancelled and that (10b) has only a double singularity, because now both $k$ and $k'$ are controlling momenta.

The other double singularity which remains comes from the second term in the numerator of equation (10a). However we find that these quadratic singularities cancel each other, leaving us with contributions to $\sigma$ which have only single poles.

Next we consider the contribution of graph (c), where $K < \Delta$, but $K'$ is not limited. Performing the $k_0'$ integration using the Cauchy theorem we find:

$$\sigma^{(c)} = \frac{g^4 C_F C_A \mathrm{Tr}(\mathbb{I})}{16\,(2\pi)^{6+2\eta}} \cdot$$

$$\cdot \left[ -\int \frac{d^{3+\eta}\vec{k}}{K^3} \int \frac{d^{3+\eta}\vec{k}'}{(K')^3} \frac{\beta(2Kx + K'x')}{(K+K')(1-\beta x')} \frac{1}{1-y} + \right.$$

$$+ \int \frac{d^{3+\eta}\vec{k}}{K^3} \int \frac{d^{3+\eta}\vec{k}'}{(K')^3} \frac{\beta(K'x' + Kx)}{K'-K-\beta(K'x'-Kx)} \frac{1}{1-y} +$$

$$\left. +4 \int \frac{d^{3+\eta}\vec{k}}{K^2} \int \frac{d^{3+\eta}\vec{k}'}{(K')^2 - K^2} \frac{1}{|\vec{k} + \vec{k}'|^2} \frac{\beta(2Kx + K'x')}{K + \beta K'x'} \right] \tag{11}$$

Note that the first expression again exhibits a triple pole singularity. However by considering also the contribution of graph (d), one finds that it cancels exactly with this expression in the region $K+K' < \Delta$. In this way we are left effectively with an expression like the first term in equation (11), but where $K+K' > \Delta$. This yields only a single pole singularity coming from the region $y \to 1$. Similarly, in the second term of equation (11), the cubic pole contribution actually vanishes by antisymmetry in the region $K+K' < \Delta$, leaving only to a



single pole coming from the region $K+K' > \Delta$, $y \to 1$.

Finally, in the Feynman gauge, the contributions from graphs (e) and (f) vanish identically because of the antisymmetry of the 3-gluon vertex. [It is also because of this property that graph (b) yields an U.V. finite result in this gauge].

Therefore we obtain that the contributions to $\sigma$ have only single poles $\eta^{-1}$. These contributions are proportional to $\Delta^{2\eta}$ so that when applying the operation $\Delta \frac{d}{d\Delta}$ in equation (6), we obtain as expected a completely finite result for the factor G. Letting now $\eta = 0$, we obtain after a very long calculation that the contribution of the non-abelian graphs is:

$$G^{(4)}_{na} = \frac{g^4 C_F C_A \mathrm{Tr}(\mathbb{I})}{(2\pi)^4} \left\{ \frac{1}{2\beta} \left[ \mathrm{Li}_3\left(\frac{1-\beta}{1+\beta}\right) - \mathrm{Li}_3\left(\frac{1+\beta}{1-\beta}\right) \right] \right.$$

$$- \frac{3}{2} \left[ \mathrm{Li}_2\left(\frac{2\beta}{1+\beta}\right) + \mathrm{Li}_2\left(\frac{-2\beta}{1-\beta}\right) \right] + \frac{1}{4\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \left[ \mathrm{Li}_2\left(\frac{1+\beta}{1-\beta}\right) + \mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right) \right]$$

$$- \frac{1}{2} \ln^2\left(\frac{1+\beta}{1-\beta}\right) + \left(\frac{\pi^2}{3} - 1\right) \left[ \frac{1}{\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 2 \right] - \left[ \frac{\pi^2}{12} - 1 + \ln 2 \right] \frac{1}{\beta} \ln\left(\frac{1+\beta}{1-\beta}\right)$$

$$+ \frac{1+\beta}{2\beta} \left[ \ln\left(\frac{1+\beta}{\beta}\right) \ln\left(\frac{1+\beta}{1-\beta}\right) + \mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right) - \frac{\pi^2}{6} \right] +$$

$$\left. + \frac{1-\beta}{2\beta} \left[ \ln\left(\frac{1-\beta}{\beta}\right) \ln\left(\frac{1+\beta}{1-\beta}\right) + \mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right) + \frac{1}{2} \ln^2\left(\frac{1+\beta}{1-\beta}\right) - \frac{\pi^2}{2} \right] \right\} \tag{12}$$

Again, analogously to equation (8), we see that in the high energy limit, when $\beta \to 1$, the above expression yields contributions proportional to $\ln^3(1-\beta)$.

Finally, we show in Figure 4 the graphs which contribute to the effective coupling constant $\bar{g}^2(\frac{\Delta}{\mu})$ for the pure Yang-Mills theory as defined by the renormalization group

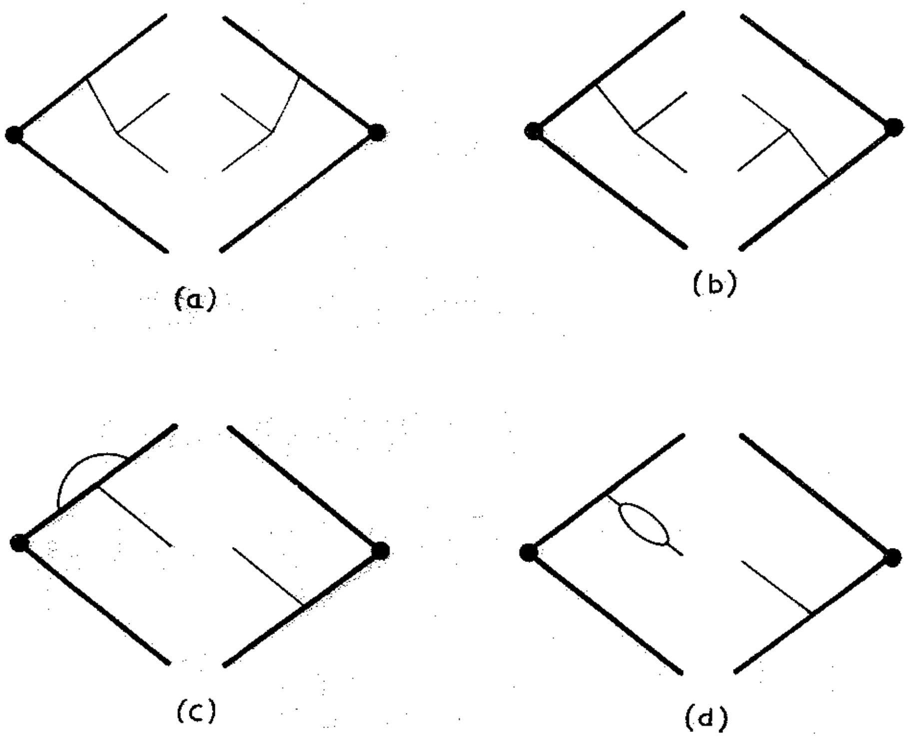


Fig.4.

In these graphs the contribution from the ghost particles is to be understood. In appendix B we present a more detailed discussion of their contribution. We obtain from the



above renormalization-group graphs the result:

$$G_{r}^{(4)}\left(\beta,\frac{\Delta}{\mu}\right)=\frac{g^4 C_F C_A T_R(I)}{(2\pi)^4}\left\{\frac{7}{3}-\frac{5}{12\beta}\left[Li_2\left(\frac{-2\beta}{1-\beta}\right)-Li_2\left(\frac{2\beta}{1+\beta}\right)\right]\right.$$

$$\left.+\left[\frac{53}{18}+\frac{11}{6}\left(\frac{1}{2}\ln(\pi)-\frac{\gamma}{2}-\ln\left(\frac{\Delta}{\mu}\right)\right)\right]\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2\right]\right\} \tag{13}$$

where $\gamma$ stands for the Euler constant.

We now add all the contributions to $G^{(4)}$ resulting from fourth order diagrams. From equations (8), (12) and (13) we find that

$$G^{(4)}=\frac{g^4 C_F C_A T_R(I)}{(2\pi)^4}\left\{\left[\frac{1}{\beta^2}-\frac{5}{12\beta}-\frac{3}{2}\right]Li_2\left(\frac{-2\beta}{1-\beta}\right)+\frac{10}{3}-\frac{(3\beta+1)\pi^2}{12\beta}\right.$$

$$+\left[\frac{\pi^2}{2}\left(\frac{1}{2}-\frac{1}{\beta}\right)+\frac{53}{18}+\frac{11}{6}\left(\frac{1}{2}\ln(\pi)-\frac{\gamma}{2}-\ln\left(\frac{\Delta}{\mu}\right)\right)\right]\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2\right]$$

$$\left.+\left[\frac{1}{\beta^2}+\frac{5}{12\beta}-\frac{3}{2}\right]Li_2\left(\frac{2\beta}{1+\beta}\right)-\frac{1}{\beta^2}Li_3(1)+O(1-\beta)\right\} \tag{14a}$$

with $O(1-\beta)$ representing a contribution which vanishes as $\beta \to 1$ (see appendix A) as follows:

$$O(1-\beta)=\frac{1}{2\beta}\left[\frac{(1-\beta)}{\beta}Li_3\left(\frac{1+\beta}{1-\beta}\right)+\frac{(1+\beta)}{\beta}Li_3\left(\frac{1-\beta}{1+\beta}\right)\right]+\frac{1}{2}\frac{(1-\beta^2)}{\beta^2}\ln^2\left(\frac{1+\beta}{1-\beta}\right)$$

$$-\frac{1}{4\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\left[\frac{(1-\beta)}{\beta}Li_2\left(\frac{1+\beta}{1-\beta}\right)-\frac{(1+\beta)}{\beta}Li_2\left(\frac{1-\beta}{1+\beta}\right)\right]+\frac{(1+\beta)}{2\beta}Li_2\left(\frac{1-\beta}{1+\beta}\right)$$

$$+\frac{(1-\beta)}{2\beta}\left[\ln\left(\frac{1-\beta}{\beta}\right)\ln\left(\frac{1+\beta}{1-\beta}\right)+Li_2\left(\frac{1-\beta}{1+\beta}\right)+\frac{1}{2}\ln^2\left(\frac{1+\beta}{1-\beta}\right)-\frac{\pi^2}{2}\right]$$

$$+\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\left[\frac{(1+\beta)}{2}\ln\left(\frac{1+\beta}{\beta}\right)-\ln(2)\right] \tag{14b}$$

It is interesting to remark that in the final result, the contributions which potentially could yield at high energies $\ln^3(1-\beta)$ terms, are suppressed by a factor $(1-\beta)$. The physical reason for this behaviour will be given in the next section.

## III. DISCUSSION

From the results obtained in the preceding section [equation (7b) and (14)] we can express the factor $G\left(\frac{m}{\sqrt{s}},\frac{\Delta}{\mu}\right)$ in the following form:

$$G\left(\frac{m}{\sqrt{s}},\frac{\Delta}{\mu}\right)=\frac{C_F\,\mathrm{Tr}(I)}{(2\pi)^2}\left\{g^2\left[1-\frac{11C_A}{24\pi^2}g^2\ln\left(\frac{m\Delta}{\sqrt{s}\,\mu}\right)\right]B\left(\frac{m}{\sqrt{s}}\right)\right.$$

$$+\frac{g^4C_A}{(2\pi)^2}\left[-\frac{23}{9}+\frac{11}{6}\left(\frac{\pi^2}{6}-\ln(\pi)+\gamma\right)-Li_3(1)\right]+$$



$$\left.+\frac{g^4C_A}{(2\pi)^2}\left[-\frac{139}{9}+\pi^2+\frac{11}{3}\left(\gamma-\ln(\pi)\right)\right]\ln\left(\frac{m}{\sqrt{s}}\right)+\tilde{O}\left(\frac{m^4}{s^2}\right)\right\} \tag{15}$$

Here $B\left(\frac{m}{\sqrt{s}}\right)$ denotes the bremsstrahlung function and $\tilde{O}\left(\frac{m^4}{s^2}\right)$ is related to the function $O(1-\beta)$ defined in equation (14b).

We notice that the expression in the first bracket of equation (15) is the beginning for the effective coupling constant $g^2\left(\frac{m}{\sqrt{s}}\,\frac{\Delta}{\mu}\right)$. It is no accident that the coefficient $\frac{11C_A}{24\pi^2}\ln\left(\frac{\Delta}{\mu}\right)$ appears in the above expression. This factor comes from same vertex and self-energy functions as it does in the renormalization group equations and indeed it should be present to all orders in perturbation theory [10,11].

With regard to the factor $\frac{11C_A}{24\pi^2}\ln\left(\frac{m}{\sqrt{s}}\right)$, note that it is connected, in the limit $\beta\to 1$, with the expression multiplying $Li_2\left(-\frac{2\beta}{1-\beta}\right)$ in equation (14a). We can trace back its appearance from contributions resulting from graphs with two real gluons, like the ones shown in figures 2, 3 and 4. Since this set yields a gauge invariant answer, the above result is more clearly understood in physical gauges (like the axial or Coulomb gauges). In such gauges, the mass singularities are connected with configurations where gluons are nearly parallel to a given quark line. The leading singularities result when all gluons are simultaneously parallel to a given quark line. We can estimate the degree of divergence of these singularities, by observing that any 3-point vertices vanish like the angle between two particles in a quasi-parallel configuration, by helicity conservation. A power counting analysis, similar to that in reference [12] indicates that graphs like (2a) and (3e) will yield only single logarithmic singularities, whereas the graph (4a) is the only one giving a double mass singularity. [This explains, using the gauge invariance of our result, why the cubic mass singularities proportional to $\ell n^3(1-\beta) - \ell n^3\left(\frac{m}{\sqrt{S}}\right)$ cancel in the Feynman gauge as $\beta \to 1$]. In a physical gauge [13] this graph yields precisely the factor $\frac{11C_A g^2}{24\pi^2}\,\ell n\left(\frac{m}{\sqrt{S}}\right)$, connected with the running coupling constant $\bar{g}^2\left(\frac{m}{\sqrt{S}}\right)$ in the leading approximation.

To higher orders, we therefore expect that the leading mass singularities $\left[g^2\ \ell n\ \frac{m}{\sqrt{S}}\right]^n$ to result, in the CM system, from graphs like the one shown in Figure 5, where all gluons are in a parallel configuration with one or another quark line.

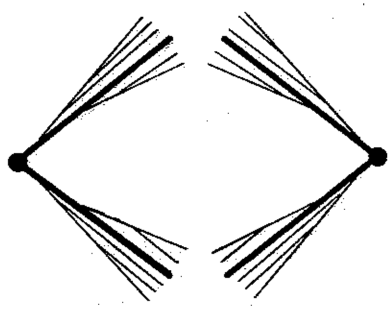


Fig. 5.

Consequently we conjecture that the factor G might have in the high-energy regime the following general form [compare with equation (15)]:



$$G\left(\frac{m}{\sqrt{S}},\frac{\Delta}{\mu}\right) = \frac{C_F\,\mathrm{Tr}(\mathbb{1})}{(2\pi)^2}\left\{\bar{g}^2\left(\frac{m\Delta}{\sqrt{S}\mu}\right)B\left(\frac{m}{\sqrt{S}}\right) + \sum_{n=2}^{\infty}\sum_{n'=0}^{n-1} C_{n,n'}\,\bar{g}^{2n}\left(\frac{\Delta}{\mu}\right)\left[\ell n\left(\frac{m}{\sqrt{S}}\right)\right]^{n'} + O\left(\frac{m^4}{S^2}\right)\right\} \tag{16a}$$

where $\bar{g}^2(t)$ is the effective coupling constant as given in the leading approximation by the renormalization group:

$$\bar{g}^2(t) = g^2\left[1+\frac{11}{24\pi^2}C_A\,g^2\ell n(t)\right]^{-1} \tag{16b}$$

The important feature of this equation is that the leading behaviour in $\frac{m}{\sqrt{S}}$ is contained in the first term where the running coupling constant multiplies the bremsstrahlung function $B\left(\frac{m}{\sqrt{S}}\right) = 4\ \ell n\left(\frac{\sqrt{S}}{m}\right) - 2$

## IV. CONCLUSION

We must now relate the previous results with the cross section $\sigma_{ph}$ defined in equation (3). To this end, we take on both sides of this equation the derivative with respect to $\Delta$. Using (6), we find:

$$\bar{\sigma}(y) = \sigma(0) + \int_0^\infty \frac{d\Delta'}{\Delta'}\,G\left(\frac{\Delta'}{\mu}\right)e^{-i\Delta' y} \tag{17a}$$

To fourth order, $G(\frac{\Delta}{\mu})$ has the form $a - b\, \ell n \frac{\Delta}{\mu}$, where a and b are determined by (15). In particular we have:

$$b = \frac{11 C_F C_A Tr(I)}{96\pi^4} g^4 B\left(\frac{m}{\sqrt{s}}\right) \tag{17b}$$

Using (17) to order $g^4$ for the non-abelian terms proportional to $C_F C_A$, we obtain with the help of equation (4a) the result:

$$\bar{G}(y\mu) = -\left[a - b \ln\left(\frac{1}{iy\mu}\right)\right] - b\gamma \tag{18}$$

which yields the structure indicated in equation (5).

We proceed to calculate the factor Y [see (3b) and (4b)]. To this end we recall that $\bar{\sigma}(y)$ is an IR finite quantity, so that the contributions from the purely virtual graphs contained in $\sigma(0)$ must cancel the infrared divergences present in the second term of equation (17a). Since $\sigma(0)$ is a real quantity, independent of $\Delta$ and $y$, the virtual graphs must provide effectively a subtraction at a point $y_0 = \frac{1}{iM}$ where M is some parameter with dimension of mass. Using (18) and (4b), we then find:

$$Y(y) = -\left[a + b\gamma + \frac{b}{2} \ln\left(\frac{iy\mu^2}{M}\right)\right] \ln(iyM) \tag{19}$$

The fact that $Y^*(-y) = Y(y)$ guarantees the reality of the cross section $\sigma_{ph}$ defined in (3). Introducing a dimensionless variable $z = \Delta y$, and using (19) we obtain from this equation the result:



$$\sigma_{ph} = |f|^2 \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dz}{z - i\epsilon}\, e^{iz} \cdot$$

$$\cdot \exp - \left\{\left[a + b\gamma + \frac{1}{2} b \ln\left(\frac{iz\mu^2}{\Delta M}\right)\right] \ln\left(\frac{izM}{\Delta}\right)\right\} \tag{20}$$

In order to evaluate this integral, it turns out to be more convenient to express it in terms of G and b, rather than as a function of a and b. So, after factorizing an exponential term independent of z, we can write (20) in the form:

$$\sigma_{ph} = |f|^2 e^{-\left\{\left[G\left(\frac{m}{\sqrt{s}}, \frac{\sqrt{\Delta M}}{\mu}\right) + b\gamma\right] \ln\left(\frac{M}{\Delta}\right)\right\}}\, I\left(\frac{m}{\sqrt{s}}, \frac{\Delta}{\mu}\right) \tag{21a}$$

where $I\left(\frac{m}{\sqrt{s}}, \frac{\Delta}{\mu}\right)$ is determined by the integral:

$$I\left(\frac{m}{\sqrt{s}}, \frac{\Delta}{\mu}\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dz}{iz + \epsilon}\, e^{iz} \cdot \tag{21b}$$

$$\cdot \exp - \left\{\left[G\left(\frac{m}{\sqrt{s}}, \frac{\Delta}{\mu}\right) + b\gamma + \frac{1}{2} b \ln(iz)\right] \ln(iz)\right\}$$

In this expression the function G is given by equation (15), while b is determined by (17b). From these equations we see that in the high energy limit, we have $G \gg b \gg 1$. This is because, to the order we are working, G contains entirelly all the leading dependence, whereas b is non-leading. We can then evaluate the integral using the saddle point method. Neglecting terms of order $[b/G]^2$, we find in this way that:

$$I \simeq \frac{1}{\sqrt{2\pi}}\ \frac{e^G}{G^{(G-1/2)}} \cdot \tag{22a}$$

$$\cdot e^{-b\ln(G)[\frac{1}{2}\ln(G)+\gamma]}\left\{1+\frac{b}{2G}\left[\ln(G)+1+\gamma\right]\right\}$$

The first factor in this expression yields precisely the asymptotic form of $[\Gamma(G)]^{-1}$, where $\Gamma$ stands for the gamma function. In fact, if $b=0$, we can evaluate exactly the integral in (21b) and obtain [14]:

$$I(b=0)=\left[\Gamma(G+1)\right]^{-1} \tag{22b}$$

This case is relevant in QED, where $C_A=0$ so that $b$ vanishes.

Equations (21) and (22) represent our result which gives the leading and non-leading logarithms of $\frac{m}{\sqrt{s}}$ and $\frac{\Delta}{\mu}$ as well as the finite contributions to the cross section. It is interesting to remark that classically, we expect the angular width of a jet made of a quark and gluons to be of order $\frac{1}{2}\sqrt{1-\beta_{CM}^2} = \frac{m}{\sqrt{s}}$ in the center of mass frame. So it is plausible to interpret $\frac{m}{\sqrt{s}}$ and $\frac{\Delta}{\mu}$ as representing the angular and energy resolutions of QCD jets, respectively. Our expression for $\sigma_{ph}$ shows that the cross section decreases very rapidly as these parameters are scaled to zero. This represents a sensible behaviour, in contrast to the low orders QCD contributions which diverge in this limit.



APPENDIX A

We present here a list of formulae [8], which we found particularly useful for the calculations encountered in the evaluation of the $4^{th}$ order diagrams. The dilogarithm $Li_2(x)$ is defined by:

$$Li_2(x) = -\int_0^x \frac{\ln(1-z)}{z}\,dz \tag{A.1}$$

and has the properties:

$$Li_2(1)=\pi^2/6 \ ; \qquad Li_2(-1)=-\pi^2/12 \tag{A.2}$$

$$Li_2(-x)+Li_2(-1/x) = -\pi^2/6 - \frac{1}{2}\ln^2(x) \quad , x>0 \tag{A.3}$$

$$Li_2(x)+Li_2(1/x) = \pi^2/3 - \frac{1}{2}\ln^2(x) - i\pi\ln(x) \quad , x>1 \tag{A.4}$$

The trilogarithm $Li_3$ is defined by:

$$Li_3(x) = \int_0^x \frac{Li_2(z)}{z}\,dz \tag{A.5}$$

and has the properties ($\zeta$ is the zeta function of Riemann):

$$Li_3(-1) = \frac{-3}{4}\,Li_3(1) = \frac{-3}{4}\zeta \tag{A.6}$$

$$Li_3(-x)-Li_3(-1/x) = \frac{-\pi^2}{6}\ln(x) - \frac{1}{6}\ln^3(x) \quad , x>0 \tag{A.7}$$

$$Li_3(x)-Li_3(1/x) = \frac{\pi^2}{3}\ln(x) - \frac{1}{6}\ln^3(x) - \frac{i\pi}{2}\ln^2(x) \quad , x>1 \tag{A.8}$$

APPENDIX B

In this appendix we consider in some detail the contributions which result from the renormalization-group diagrams shown in Figure 4. After a straightforward application of the Feynman rules in the eikonal approximation, graphs (a) and (b) yield:

$$\Lambda = \frac{g^4 C_F C_A \operatorname{Tr}(I)}{8\,(2\pi)^{6+2\eta}} \int \frac{d^{3+\eta}k}{K} \int \frac{d^{3+\eta}k'}{K'} \left\{ \frac{5}{2n^4}\left[\frac{p\cdot\Delta}{p\cdot n} - \frac{q\cdot\Delta}{q\cdot n}\right]^2 + \frac{4}{n^2}\left[\frac{p^2}{(p\cdot n)^2} + \frac{q^2}{(q\cdot n)^2} - \frac{2p\cdot q}{(p\cdot n)(q\cdot n)}\right]\right\} \tag{B.1}$$

where

$$n \equiv k + k' \quad ; \quad \Delta \equiv k - k' \tag{B.2}$$

Except for the second term in second bracket above, we will perform all calculations in the rest frame of p , where in view of the condition (2a) we have $K+K' < \Delta$ . The contribution of second term just mentioned will be done in rest frame of q . It will yield, in addition to a contribution equal to that of first term, a correction $\Lambda_b$ , which results from the fact that we must boost appropriately the condition $K+K' < \Delta$ , from the rest frame of p to the rest of q . Denoting the contribution of first bracket in (B.1) by $\Lambda_1$ , the contribution of first two terms in second bracket by $\Lambda_2 + \Lambda_b$ and that of last term by $\Lambda_3$ , we can write (B.1) as follows:



$$\Lambda \equiv \frac{g^4 C_F C_A \operatorname{Tr}(I)}{8\,(2\pi)^{6+2\eta}} \left[\frac{5}{8}\beta^2 \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_b\right] \tag{B.3}$$

where:

$$\Lambda_1 = \int \frac{d^{3+\eta}k}{K} \int \frac{d^{3+\eta}k'}{K'(K+K')^2} \frac{(x-x')^2}{\left[\beta(Kx+K'x') - (K+K')\right]^2} \frac{1}{(1-y)^2} \tag{B.3a}$$

$$\Lambda_2 = \int \frac{d^{3+\eta}k}{K^2} \int \frac{d^{3+\eta}k'}{(K')^2} \frac{1}{(K+K')^2} \frac{1}{(1-y)} \tag{B.3b}$$

$$\Lambda_3 = \int \frac{d^{3+\eta}k}{K^2} \int \frac{d^{3+\eta}k'}{(K')^2(K+K')} \frac{1}{\beta(Kx+K'x') - (K+K')} \frac{1}{(1-y)} \tag{B.3c}$$

We start with the calculation of $\Lambda_1$ , which we write as a sum of two terms: one which is regular as $y \to 1$ , the other one being singular in this limit:

$$\Lambda_1 \equiv -\left(1 + \beta\frac{d}{d\beta}\right)\left(\Lambda_{11} + \Lambda_{12}\right) \tag{B.4}$$

where:

$$\Lambda_{11} = \beta \int \frac{d^{3+\eta}k}{K} \int \frac{d^{3+\eta}k'}{(K+K')^4} \frac{1}{(\beta x - 1)} \frac{1}{\beta(Kx+K'x') - (K+K')} \frac{(x-x')^3}{(1-y)^2} \tag{B.4a}$$

$$\Lambda_{12} = \int \frac{d^{3+\eta}k}{K} \int \frac{d^{3+\eta}k'}{K'(K+K')^4} \frac{1}{(\beta x - 1)} \frac{(x-x')^2}{(1-y)^2} \tag{B.4b}$$

In order to determine $\Lambda_{11}$ we use the relation $y = xx' + \sqrt{(1-x^2)(1-x'^2)}\cos\xi$ and define two parameters $\tau$ and $\lambda$ as follows:

$$K \equiv \lambda\tau \quad ; \quad K' \equiv (1-\lambda)\tau \tag{B.5a}$$

where in view of condition (2a) we have:

$$0 \leqslant \tau \leqslant \Delta \quad ; \quad 0 \leqslant \lambda \leqslant 1 \tag{B.5b}$$

The importance of this change of variables is that the $\tau$ integration factorizes and can then be easily done. After performing also the $\xi$ integration we find:

$$\Lambda_{11} = \frac{(2\pi)^{2}\pi^{-\eta}\beta}{\Gamma^2(1+\eta/2)}\,\frac{(\Delta)^{2\eta}}{(2\eta)}\int_0^1 d\lambda(\lambda)^{1+\eta}(1-\lambda)^{2+\eta}\int_{-1}^{1}dx(1-x^2)^{\eta/2}\,\cdot$$

$$\cdot\int_{-1}^{1}dx'(1-x'^2)^{\eta/2}\frac{1}{(\beta x-1)}\,\frac{(1-xx')\,\mathrm{sign}(x-x')}{\beta[\lambda x+(1-\lambda)x']-1} \tag{B.6a}$$

We are actually interested in determining $G\left(\beta,\frac{\Delta}{\mu}\right)$ defined in equation (6). Since the operation $\Delta\frac{d}{d\Delta}$ brings a factor of $\eta$ in the numerator, we can effectively put $\eta=0$ in the integrations above. In this way, performing the integration over $\lambda$ and $x'$ we find:

$$\Lambda_{11} = 4\pi^2\,\frac{\Delta^{2\eta}}{(2\eta)}\left\{\frac{1}{3}\int_{-1}^{+1}\frac{x^2}{1-\beta x}dx\; + \right.$$



$$+\int_{-1}^{1}\frac{dx}{(1-\beta x)}\left(1-\frac{x}{\beta}\right)\left[\frac{-(1-\beta)^2}{2\beta^2(1-x)^2}\ln\left(\frac{1-\beta x}{1-\beta}\right)-\frac{1}{4}+\frac{1-\beta}{2\beta(1-x)}\right]$$

$$-\int_{-1}^{1}dx\,\frac{1+x}{1-\beta x}\left[\frac{(1-\beta)^3}{3\beta^3(1-x)^2}\ln\left(\frac{1-\beta x}{1-\beta}\right)-\frac{1-x}{9}+\frac{1-\beta}{6\beta}-\frac{(1-\beta)^2}{3\beta^2(1-x)}\right]$$

$$\left. +\left[x\longrightarrow -x\,;\;\beta\longrightarrow-\beta\right]\right\} \tag{B.6b}$$

Now we turn to the calculation of $\Lambda_{12}$, which has a singularity when $y \to 1$. For this reason, it is convenient in this case to eliminate $x'$ via the relation: $x' = xy + \sqrt{(1-x^2)(1-y^2)}\cos\psi$. Performing then the $\tau, \lambda, \psi$ and $y$ integrations, using the condition (B.5), we find:

$$\Lambda_{12} = -4\pi^2\frac{\Delta^{2\eta}}{(2\eta)}\left\{\frac{1}{3}\int_{-1}^{1}\frac{x^2}{1-\beta x}dx\; + \right.$$

$$+(2\pi)^{\eta}\frac{\Gamma(2+\eta)}{\Gamma(4+2\eta)}\left(\frac{2}{\eta}+1\right)\left[\int_{-1}^{1}\frac{1-x^2}{1-\beta x}dx\; + \right.$$

$$\left.\left. +\frac{\eta}{2}\int_{-1}^{1}\frac{(1-x^2)\ln(1-x^2)}{(1-\beta x)}\right]\right\} \tag{B.7}$$

Adding now the expressions for $\Lambda_{11}$ and $\Lambda_{12}$, we observe that the first terms in (B.6b) and (B.7) cancel out. Performing finally the $x$ integration, we obtain after a long calculation the following contribution to $\Lambda_1$, as defined in equations (B.3) and (B.4):

$$\Lambda_1 = 4(2\pi)^{2+2\eta}\beta^{-2}\frac{\Delta^{2\eta}}{(2\eta)}\left\{\frac{4}{3}(2-\ln(2)) + \right.$$

$$+\frac{2}{3}\left[\frac{1}{\eta}+\gamma-\ln(2\pi)+\frac{1}{6}\right]\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2\right]+$$

$$+\frac{1}{3\beta}\left[Li_2\left(\frac{-2\beta}{1-\beta}\right)-Li_2\left(\frac{2\beta}{1+\beta}\right)-Li_2\left(\frac{1+\beta}{1-\beta}\right)+Li_2\left(\frac{1-\beta}{1+\beta}\right)\right]$$

$$\left.-\frac{1}{3\beta}\ln\left(\frac{\beta^2-1}{\beta^2}\right)\ln\left(\frac{1-\beta}{1+\beta}\right)\right\} \qquad (B.8)$$

At this point we mention that the contribution of the ghost particles, turns out be yield a result which is one fifth of the one quoted above. Proceeding analogously, we find that the contribution resulting from $\Lambda_2$ and $\Lambda_3$, as defined in (B.3) are given respectively by:

$$\Lambda_2 = 16(2\pi)^{2+2\eta}\frac{\Delta^{2\eta}}{(2\eta)}\left[\frac{1}{\eta}-\ln(2\pi)+\gamma-3+\ln(2)\right] \qquad (B.9)$$

and

$$\Lambda_3 = 16(2\pi)^{2+2\eta}\frac{\Delta^{2\eta}}{2\eta}\left\{\left(-\frac{1}{\eta}+\ln(2\pi)-\gamma+3\right)\frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)+\right.$$



$$+\frac{1}{4\beta}\left[Li_2\left(\frac{1+\beta}{1-\beta}\right)-Li_2\left(\frac{1-\beta}{1+\beta}\right)+Li_2\left(\frac{2\beta}{1+\beta}\right)-Li_2\left(\frac{-2\beta}{1-\beta}\right)\right]$$

$$\left.-\frac{1}{4\beta}\ln\left(\frac{\beta^2-1}{\beta^2}\right)\ln\left(\frac{1+\beta}{1-\beta}\right)\right\} \qquad (B.10)$$

Lastly we consider the correction $\Lambda_b$. This comes about because, in the rest frame of $q$, the condition $\tau < \Delta$ (see (B.5) must be boosted and replaced by $\tau < \sqrt{1-\beta^2}\,[\lambda(1-\beta x) + (1-\lambda)(1-\beta x')]^{-1}\Delta$. This modification turns out to yield the correction:

$$\Lambda_b = -8(2\pi)^{2+2\eta}\frac{\Delta^{2\eta}}{(2\eta)}\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2\right] \qquad (B.11)$$

Adding all contributions listed in (B.3), we obtain after applying $\Delta\frac{d}{d\Delta}$ to (B.3) the following result:

$$\Delta\frac{d\Lambda}{d\Delta} = \frac{g^4 C_F C_A \mathrm{Tr}(I)}{(2\pi)^4}\left\{\frac{2}{3}+\frac{5}{3}\ln(2)+\right.$$

$$+\frac{5}{3}\left(\frac{-1}{2\eta}+\frac{73}{60}+\frac{1}{2}\ln(2)+\frac{1}{2}\ln(\pi)-\frac{1}{2}\gamma-\ln\left(\frac{\Delta}{\mu}\right)\right)\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2\right]+$$

$$+\frac{5}{12\beta}\left[Li_2\left(\frac{1+\beta}{1-\beta}\right)-Li_2\left(\frac{1-\beta}{1+\beta}\right)+Li_2\left(\frac{2\beta}{1+\beta}\right)-Li_2\left(\frac{-2\beta}{1-\beta}\right)\right]$$

$$\left.-\frac{5}{12\beta}\ln\left(\frac{\beta^2-1}{\beta^2}\right)\ln\left(\frac{1+\beta}{1-\beta}\right)\right\} \qquad (B.12)$$

Proceeding in a similar way, we obtain for the contribution of the abelian graph (4c) the result:

$$\Delta \frac{d}{d\Delta} \Lambda_c = \frac{g^4 C_F C_A \operatorname{Tr}(I)}{(2\pi)^4} \cdot$$

$$\cdot \left[1 + \frac{1}{2} \ln(\pi) - \frac{1}{2}\gamma - \frac{1}{2} - \ln\left(\frac{\Delta}{\mu}\right)\right]\left[\frac{1}{\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 2\right] \tag{B.13}$$

Finally the non-abelian graph (4d) yields the contribution:

$$\Delta \frac{d}{d\Delta} \Lambda_d = \frac{g^4 C_F C_A \operatorname{Tr}(I)}{(2\pi)^4} \left\{ \frac{5}{3} - \frac{5}{3} \ln(2) - \right.$$

$$- \frac{5}{6} \left[ \frac{-1}{\eta} + \ln(2) + \frac{1}{2} \ln(\pi) - \frac{1}{2}\gamma - \ln\left(\frac{\Delta}{\mu}\right) - \frac{1}{2} \right] \left[ \frac{1}{\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 2 \right] +$$

$$\left. + \frac{5}{12\beta} \left[ \ln\left(\frac{\beta^2-1}{\beta^2}\right) \ln\left(\frac{1+\beta}{1-\beta}\right) - \mathrm{Li}_2\left(\frac{1+\beta}{1-\beta}\right) + \mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right) \right] \right\} \tag{B.14}$$

Adding the contributions corresponding to (B.12), (B.13) and (B.14) we find that the infrared divergent poles cancel, yielding the result stated in equation (13).



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