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ABSTRACT

This work studies the anatomy of the dynamics of quantum correlations of two interacting subsystems described by the Jaynes-Cummings Model[1], making use of a natural states decomposition, following an old suggestion by Schroedinger. The amplitude modulation of the fast Rabi oscillations which occur for a strong, coherent initial field is obtained from the spin intrinsic depolarization resulting from corrections to the mean field approximation.

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RESUMO

Este trabalho estuda a anatomia da dinâmica das correlações quânticas de dois subsistemas em interação descritos pelo Modelo de Jaynes-Cummings[1], fazendo uso de uma decomposição nos estados naturais, segundo uma antiga sugestão de Schroedinger. A modulação da amplitude das oscilações rápidas de Rabi, que ocorrem para um campo inicial intenso e coerente, é obtida da despolarização intrínseca do spin que resulta das correções à aproximação de campo médio.

INTRODUCTION

Open subsystems of even very simple, closed quantum mechanical systems can display very intrincate dynamical behavior, described by an effective, non-unitary time evolution law for the density matrix which may be used to describe their state. This fact is dearly illustrated by the spin observables of the so called coherent Jaynes-Cummings Model, which has recently been studied in considerable detail[2]. In this work, however, essential use is made of the fact that the model is soluble, albeit not in closed form, and a lot of effort is successfully spent in obtaining precise, computable approximations to the exact solutions, valid both for short and for long times. These requirements are stringent enough to place stronger demands on mathematical expediency than on physical transparency, and the present work in a way attempts at reversing this situation. We restrict ourselves, in fact, to moderately short times only (i.e., of the order of several periods of the fast Rabi oscillations, see below) and base the analysis of the dynamics on physically motivated quantities. We are able to show that the remarkable behavior of the Jaynes-Cummings Model during the times that warrant the application of our approach can be understood in terms of the behavior of very simple and straightforward spin observables, which are however governed by rather intricate laws due to the dynamical evolution of quantum correlations between the two subsystems which are involved in the model. Since we make no use of the soluble character of the model, the analysis can in principle be extended to other systems and situations.

We begin with a short characterization of the model and with the definition of the relevant observables and

parameters in Sections I and II. Ou· approach to the dynamics of subsystems, including correlations between different subsystems is described in Sections III to V. Finally, in Section VI we present numerical results and a final discussion.

I. THE MODEL

The Jaynes-Cummings Model[1] is characterized by the exactly soluble hamiltonian H , that models the interaction of the radiation with matter,

$$H = \frac{\epsilon}{2}\sigma_3 + a^+a + \lambda(a\sigma_+ + a^+\sigma_-) \tag{1}$$

where a and a^+ are bosonic operators for the annihilation and creation of photons respectively ($[a,a^+] = 1$), associated with one normal mode of the radiation field, and σ_3 , $\sigma_\pm = \frac{\sigma_1 \pm i\sigma_2}{2}$ are spin operators satisfying angular momentum commutation rules. This degree of freedom describes a two level "matter" system, and ϵ is its natural transition frequency; λ is the coupling constant that represents the strength of the interaction between matter and radiation, while the frequency of the normal mode of the quantized radiation field is taken as the unit of energy ($\hbar=1$).

We concentrate our discussion on the coherent case in which one studies the time evolution of the Jaynes-Cummings system, given the initial condition

$$|t=0\rangle = |v\rangle \otimes |+\rangle$$

where $|v\rangle$ stands for a coherent state of the radiation mode and $|+\rangle$ is an eigenstate of the spin operator σ_3. This case has been extensively studied by Narozhny et al[2]. In that work it is shown in particular that the time dependence of the atomic inversion $-\langle\sigma_3\rangle_t$ (see Figure (1)) involves at least two different characteristic times. These times are associated respectively (for strong fields or large v) with a fast oscillatory behavior of $\langle\sigma_3\rangle_t$ and with a gradual damping of these oscillations. The oscillations themselves are readily associated with the precession of the spin in the strong field of the radiation mode (Rabi oscillations), but the basic physical mechanism underlying their damping remains relatively unexplored.

We show that together with the atomic inversion ($\langle\sigma_3\rangle_t$) there is another crucial quantity that can be calculated to make clearer the particular behaviors described above. It is the intrinsic atomic inversion or $\langle\sigma_p\rangle_t$ where σ_p is the projection of the operator σ along the spin polarization axis, i.e., $\sigma_p = \dfrac{\vec{\sigma}\cdot\langle\vec{\sigma}\rangle_t}{|\langle\vec{\sigma}\rangle|_t}$. This quantity is important since it gives us the degree of intrinsic polarization of the spin as a function of time. It depends on the dynamics of quantum correlations between the two subsystems in an essential way. In particular, it is trivial to check that a mean field approximation leads to $\langle\sigma_p\rangle_t$ being independent of time.

II. PERTURBATIVE TREATMENT FOR VERY SHORT TIMES

First of all, it is interesting to consider what happens to the system (spin + field) during the first moments of interaction. This means short times in comparison with the

shorter characteristic time of the system.

In this limit, a straightforward calculation for the coherent initial condition gives

$$\langle\sigma_p\rangle_t = 1 - 2\lambda^2 t^2$$

that leads us to identifying the characteristic time

$$t_D \propto \frac{1}{\lambda} \tag{2}$$

for $\langle\sigma_3\rangle_t$ one gets, on the other hand,

$$\langle\sigma_3\rangle_t = 1 - 2\lambda^2(1+|v|^2)t^2$$

with the corresponding characteristic time

$$t_R \propto \frac{1}{\lambda|v|} \tag{3}$$

For large v and small λ these times are indicated quantitatively in Figure 1. The characteristic time t_R is associated with the precession of the spin caused by the strong radiation field. As $\langle\sigma_p\rangle_t$ is a kind of measure of the depolarization of the spin, the time t_D characterizes a temporal scale at which the initial state of the spin relaxes to a non-polarized state since, moreover, t_D is of the order of the times associated with the modulation of the Rabi oscillations, one is led to associate the latter effect to the depolarization of the spin. This will in fact be investigated quantitatively in the following.

III. THE DYNAMICS OF THE SUBSYSTEMS

The system characterized by H consists of two subsystems (spin + field) interacting through the last term. This leads us naturally to consider the state-vector space for the entire quantum system as the tensor product of a "spin" space $\mathcal{H}_S$ and a "field" space $\mathcal{H}_\Omega$

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_\Omega$$

and we can say that any state vector ($|t\rangle$) contained in this tensor product of two Hilbert spaces can be expanded as[3,4,5]

$$|t\rangle = \sum_{i=1}^{2} \alpha_i(t)\,|\Omega_i(t)\rangle\,|S_i(t)\rangle \qquad (4)$$

where $\alpha_i(t)$ are real amplitudes and $\{|S_i(t)\rangle\}$ and $\{|\Omega_i(t)\rangle\}$, the natural states, are sets of orthonormal vectors in the two level system space ($\mathcal{H}_S$) and in the normal mode of the radiation field space ($\mathcal{H}_\Omega$) respectively. These sets may always be completed to form basis sets in these spaces.

As this expansion gives the density matrices of the two subsystems in diagonal form

$$\rho_S = \mathrm{tr}_\Omega\,|t\rangle\langle t| = \sum_{i=1}^{2} |\Omega_i(t)\rangle\,\alpha_i^2(t)\,\langle\Omega_i(t)|$$

$$\rho_\Omega = \mathrm{tr}_S\,|t\rangle\langle t| = \sum_{i=1}^{2} |S_i(t)\rangle\,\alpha_i^2(t)\,\langle S_i(t)|$$

we see that it can be characterized also as an expansion of the state vector that describes the entire system in the eigenvectors of the reduced density matrices for <u>each</u> subsystem.

Expanding the state vector $|t\rangle$ that describes

the entire system as in (4), we can analyze the temporal evolution of the system in terms of that of the natural states <u>and</u> of the respective occupation amplitudes $\alpha_i(t)$.

If we consider the time dependent Schroedinger equation

$$i\frac{\partial}{\partial t}|t\rangle = H|t\rangle$$

and calculate it using (4) we get

$$\dot\alpha_m(t) = \mathrm{Im}\left\{\sum_{i=1}^{2}\alpha_i(t)\langle\Omega_m(t)\,S_m(t)\,|H|\,\Omega_i(t)\,S_i(t)\rangle\right\} \qquad (5)$$

$$\alpha_m(t)\Big[\langle S_m(t)|h_S(t)|S_m(t)\rangle + \langle\Omega_m(t)|h_\Omega(t)|\Omega_m(t)\rangle\Big] = \mathrm{Re}\left\{\sum_{i=1}^{2}\alpha_i(t)\langle\Omega_m(t)\,S_m(t)\,|H|\,\Omega_i(t)\,S_i(t)\rangle\right\} \qquad (6)$$

and, for $m \neq p$,

$$(\alpha_m(t)\pm\alpha_p(t))\Big\{\langle\Omega_p(t)|h_\Omega(t)|\Omega_m(t)\rangle \pm \langle S_m(t)|h_S(t)|S_p(t)\rangle\Big\} = \sum_{i=1}^{2}\alpha_i(t)\Big\{\langle\Omega_p(t)S_m(t)|H|\Omega_i(t)S_i(t)\rangle \pm \langle\Omega_i(t)S_i(t)|H|\Omega_m(t)S_p(t)\rangle\Big\} \qquad (7)$$

where $h_\Omega(t)$ and $h_S(t)$ are two hermitean time displacement generators, acting respectively in $\mathcal{H}_\Omega$ and in $\mathcal{H}_S$. They describe the time dependence of the natural states through

$$i\frac{d}{dt}|S_i(t)\rangle = h_S(t)\,|S_i(t)\rangle$$

and

$$i\frac{d}{dt}\,|\Omega_i(t)\rangle = h_\Omega(t)\,|\Omega_i(t)\rangle$$

These operators are sufficiently defined by equations (6) and (7). It is clear thus that equations (5), (6) and (7) determine completely the dynamics of the system and they allow us to analyze conveniently the temporal evolution of each subsystem and of their mutual correlations. This will be done next.

IV. THE MEAN FIELD APPROXIMATION

A mean field approximation for coherent initial conditions is easily motivated by noting that, for small coupling λ, the envelope of the Rabi oscillations (associated with t_R, see Figure 1), a smooth quantity, together with $\langle\sigma_p\rangle_t$ remains close to 1 for several Rabi periods. This suggests the validity, for such intervals of time, of an ansatz of the form

$$|t\rangle = |\Omega_1(t)\rangle\,|S_1(t)\rangle \tag{8}$$

for the state vector of the composite system. In terms of the analysis of the preceding section, this implies constant occupation amplitude, i.e., $\alpha_i(t) = 0$ and $\alpha_1(t) = 1$. The replacement of equation (4) by these constraints, together with equation (8) and equations (5) and (6) define our mean field approximation. In fact, when we substitute the expression (8) into the coupled equations (5), (6) and (7) we get, for $h_S(t)$ and $h_\Omega(t)$, the expressions

$$h_S(t) = \langle\Omega_1(t)|H|\Omega_1(t)\rangle$$

and

$$h_\Omega(t) = \langle S_1(t)|H|S_1(t)\rangle$$

These expressions show that the generator of the temporal evolution of each subsystem is given by the average of the hamiltonian H, calculated at the state of the other subsystem. We have thus, in this approximation,

$$\begin{cases} i\frac{d}{dt}|S_1(t)\rangle = \langle\Omega_1(t)|H|\Omega_1(t)\rangle\,|S_1(t)\rangle \\ i\frac{d}{dt}|\Omega_1(t)\rangle = \langle S_1(t)|H|S_1(t)\rangle\,|\Omega_1(t)\rangle \end{cases}$$

which has to be solved for the initial condition

$$|\Omega_1(0)\rangle = |v\rangle \quad , \quad |S_1(0)\rangle = |+\rangle$$

To solve the system above we can make ansatze for the form of the states $|\Omega_1(t)\rangle$ and $|S_1(t)\rangle$

$$|\Omega_1(t)\rangle = e^{-i\varphi(t) - \frac{|v(t)|^2}{2} + v(t)a^+}\,|0\rangle \tag{9}$$

and

$$|S_1(t)\rangle = \frac{e^{-i\psi(t) + z(t)\sigma_-}}{\sqrt{1+|z(t)|^2}}\,|+\rangle \tag{10}$$

obtaining a new linear system for the parameters $v(t)$, $z(t)$, $\varphi(t)$ and $\psi(t)$ that can be conveniently dealt with numerically.

Given this mean field solution and in preparation for a perturbative evaluation of the time-dependent occupation amplitudes $\alpha(t)$, we can also determine the relevant null occupation state $|\Omega_2(t)S_2(t)\rangle$ $(\alpha_2(0)=0)$ as

$$|\Omega_2(t)S_2(t)\rangle = N e^{i\xi(t)}\left(1-|S_1(t)\rangle\langle S_1(t)|\right)\left(1-|\Omega_1(t)\rangle\langle\Omega_1(t)|\right) H\,|\Omega_1(t)S_1(t)\rangle$$

which is just the "doorway" fed by the complete hamiltonian H when it acts on $|\Omega_1(t)S_1(t)\rangle$ [6].

Using the forms (9) and (10)

$$|\Omega_2(t)S_2(t)\rangle = \frac{e^{i\xi(t)-\frac{|v(t)|^2}{2}}}{\sqrt{1+|z(t)|^2}}\left[(a^+ - v^*(t))\,e^{v(t)a^+}|0\rangle\right]\otimes\left(|-\rangle - z^*(t)|+\rangle\right)$$

which involves the parameters already obtained in (9) and (10) and the new phase $\xi(t)$.

V. THE PERTURBATIVE CORRELATION CORRECTION

To obtain our mean field approximation, we kept frozen the amplitudes $\alpha_i(t)$ imposing that $\alpha_i(t)=0$ always. To correct this, we will now allow for changes of the $\alpha_i(t)$ making use of (5) written more explicitly as

$$\dot{\alpha}_1(t) = \alpha_2(t)\,\mathrm{Im}\left\{\langle\Omega_1(t)S_1(t)|H|\Omega_2(t)S_2(t)\rangle\right\}$$

$$\dot{\alpha}_2(t) = \alpha_1(t)\,\mathrm{Im}\left\{\langle\Omega_2(t)S_2(t)|H|\Omega_1(t)S_1(t)\rangle\right\}$$

which are two coupled first order equations for $\alpha_1(t)$ and $\alpha_2(t)$. The matrix elements on the right hand side of the equations will be calculated using the states $|\Omega_1(t)\rangle|S_1(t)\rangle$ and $|\Omega_2(t)\rangle|S_2(t)\rangle$ given by the mean field solution and in this sense this is a kind of perturbative correction. Concerning the phases of the states $|\Omega_1(t)\rangle|S_1(t)\rangle$ and $|\Omega_2(t)\rangle|S_2(t)\rangle$, it turns out that, if we consider a global phase $F(t)$ given by $F(t) = \varphi(t) + \psi(t) + \xi(t)$, it is easy to prove that, for our initial conditions it remains constant $(=\frac{\pi}{2})$ [7].

On the other hand, it is necessary to keep in mind that this perturbative correction will be valid just for times short in comparison with t_D (the envoltory characteristic time) but that may be long (for $|v| >> 1$) in comparison with t_R (the Rabi characteristic time).

This gives us two coupled first order equations for $\alpha_1(t)$ and $\alpha_2(t)$ and through them we obtain the approximate temporal behavior of $\alpha_1(t)$ and $\alpha_2(t)$.

VI. RESULTS AND DISCUSSION

We concentrate on the most relevant quantities associated with the spin ("matter") system: $\langle\sigma_3\rangle_t$ – the atomic inversion and $\langle\sigma_p\rangle$ – the intrinsic atomic inversion. In terms of the parameters introduced in equations (9) and (10), they are given by

$$\langle\sigma_3\rangle_t = \left(\alpha_1^2(t) - \alpha_2^2(t)\right)\left(\frac{1-|z(t)|^2}{1+|z(t)|^2}\right)$$

and

$$\langle\sigma_p\rangle_t = \alpha_1^2(t) - \alpha_2^2(t)$$

These quantities, as obtained from numerical solution to the dynamical equations, are shown in Figure 2. We see that in this approximation $\langle\sigma_3\rangle_t$ can be written as steady Rabi oscillations modulated by a depolarization envelope which results from the dynamical correlations arising between the two subsystems.

Unlike in previous approaches (see (2) and references therein), the present approach allowed for a physical definition and for a separate calculation of the envelope, albeit in a perturbative context as far as the correlations between sub-systems are concerned. It is interesting to note that this envelope is not smooth in the sense that it contains some structure on the time scale of the Rabi oscillations. The intrinsic depolarization which goes along with the several Rabi periods following the initial time appears therefore to feel the Rabi oscillations themselves. This is not particularly surprising in view of the structure of equation (5), which relates the time evolution of the occupation amplitudes $\alpha_i(t)$ (associated with the dynamics of correlations between sub-systems) to the time dependent natural orbitals which, in particular, carry the Rabi oscillations.

A perhaps less expected result is that the average gross-structure of the depolarization envelope bears witness to the perturbative time t_D, equation (2), which was calculated at $t=0$, where the curvature of the envelope would appear naively to be possibly affected by the Rabi frequency component.

We finally stress the generality of the present approach and its ability to get hold of some rather subtle

quantum mechanical effects in a physically very transparent way. These features should stimulate its use in the analysis of other composite systems[6], particularly in the context of obtaining corrections to mean-field approximations dynamical equations.

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FIGURES CAPTION

FIG. 1 - Time Dependence of the inversion $\langle\sigma_3\rangle_t$ as a function of time, for $|v|^2 \gg 1$, showing the characteristic times t_R and t_D (just schematic).

FIG. 2 - Time Dependence of the inversion $\langle\sigma_3\rangle_t$ (smooth line) and of $\langle\sigma_p\rangle_t$ (dot line) as a function of time, for $|v|^2 \gg 1$, obtained from the numerical solution to the dynamical equations. Both of the quantities were calculated at the same points.

FIG. 1

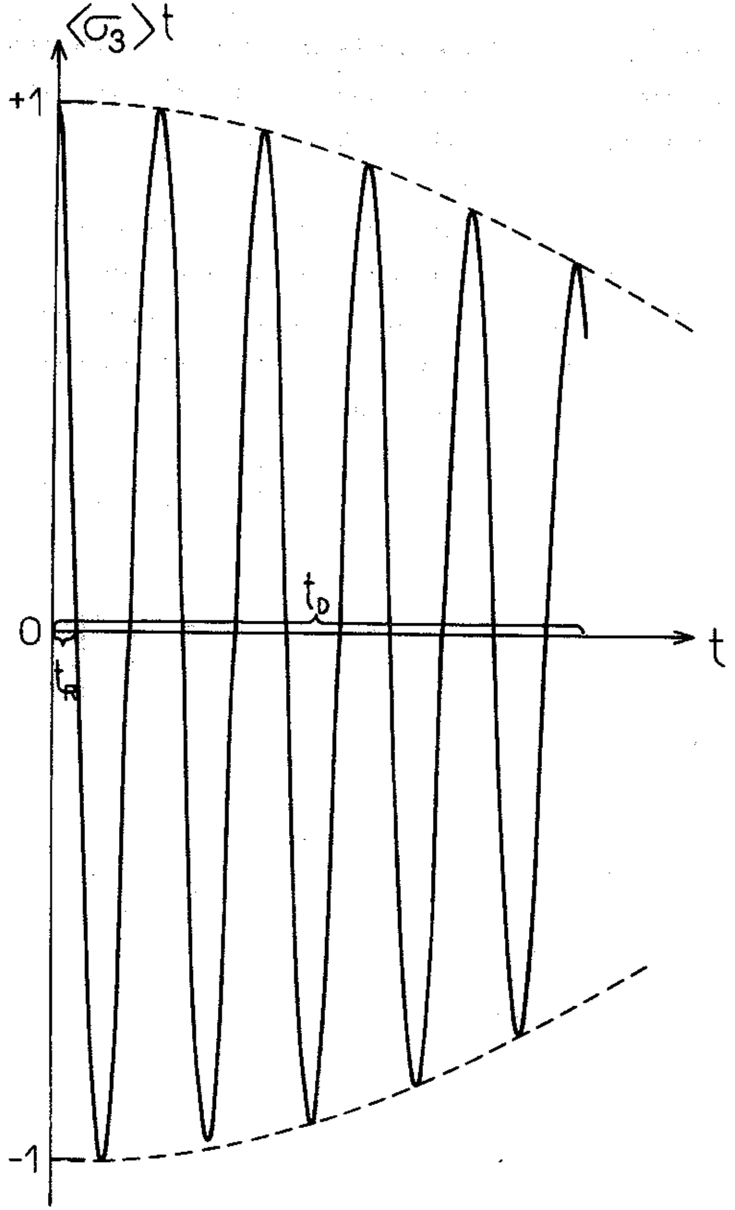

FIG. 2