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# ON THE BEHAVIOUR OF THE TEMPERATURE DEPENDENT COUPLING CONSTANTS

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TEMPERATURE DEPENDENT COUPLING CONSTANTS

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ABSTRACT

We study the temperature dependence of the coupling constant for the massive $\lambda\phi^4$ model. Using the Dyson-Schwinger equations at finite temperature we obtain the behaviour of the coupling constant in the high temperature limit. We compare our results with the ones from the one-loop approximation and the renormalization group method and conclude that, due to the breakdown of perturbation theory at high temperatures, we must improve the evaluation of the $\bar{\beta}$ function of the renormalization group in this limit.

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The dependence of the running coupling constants with the energy scale is a well stablished fact in Field Theory through the use of the renormalization group techniques[1]. Many important features of a theory can be infered from the knowledge of such dependence. For instance, by looking at the asymptotic behaviour of the effective coupling constant one can find out whether or not a given theory exhibits asymptotic freedom.

Field Theory at Finite Temperature has an extra parameter with energy dimension: the temperature (T). Then, it is expected that physical observables will depend on the temperature. This note deals with the temperature behaviour of the coupling constants in Field Theory at Finite Temperature.

One possible application of the temperature dependent coupling constants is in improving the effective potential computed in the semiclassical approximation. This improvement is achieved by replacing the zero temperature coupling constants by their running values[2,3].

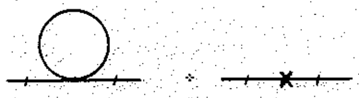
In order to fix the ideas we will consider the massive $\lambda\phi^4$ model whose euclidian lagrangian density is given by*

$$L = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2\phi^2}{2} + \frac{\lambda}{4!}\phi^4 \quad . \qquad (1)$$

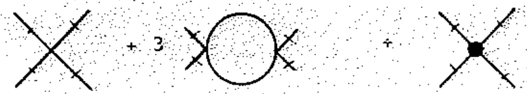
When we analyze this system at finite temperature, we obtain that its mass $(m^2(T))$ and its coupling constant

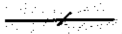
*We are assuming that $\lambda \ll 1$.

$(\lambda(T))$ are temperature dependent. In the one-loop approximation we can write[4]

$m^2(T) = m^2 -$  (2)

and

$-\lambda(T) =$  (3)

where all the external lines in (2) and (3) are taken at zero momentum,  stands for amputation of external legs, and x and ● for the counterterms needed to renormalize the mass and the coupling constant. These counterterms are fixed by imposing the following renormalization conditions for the two and four point 1PI Green functions at zero external momentum and at zero temperature

$$\Gamma^2(P_i=0\,,\,T=0) = m^2 \tag{4}$$

$$\Gamma^4(P_i=0\,,\,T=0) = -\lambda \quad . \tag{5}$$

Using the finite temperature Feynman rules, we can compute $m^2(T)$ and $\lambda(T)$ (given by (2) and (3) respectively) which satisfy (4) and (5), and we obtain

$$m^2(T) = m^2 + \frac{\lambda}{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{\sqrt{\vec{k}^2+m^2}\,\left(\exp(\beta\sqrt{\vec{k}^2+m^2}\,)-1\right)} \tag{6}$$

and



$$\lambda(T) = \lambda + \frac{3\lambda^2}{4\pi^2}\int_0^\infty dx\, x^2\,\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+m^2\beta^2}\,\left(\exp\sqrt{x^2+m^2\beta^2}-1\right)}\right) \quad . \tag{7}$$

In the high temperature limit (i.e., $T \gg m$) the asymptotic behaviour of $m^2(T)$ and $\lambda(T)$ are[5]

$$m^2(T) = m^2 + \frac{\lambda T^2}{24} \tag{8}$$

$$\lambda(T) = \lambda - \frac{3\lambda^2}{8\pi^2}\left[\frac{\pi T}{2m} + \frac{1}{2}\,\ell n\left(\frac{m}{4\pi T}\right)\right] \quad . \tag{9}$$

The above calculations can be improved by using the renormalization group at finite temperature[4,6]. The equation governing the evolution of $\lambda$ is[4]

$$\frac{d\lambda}{dt} = \bar{\beta} \tag{10}$$

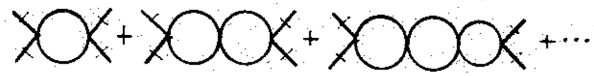
where $t = \ell n(T/m)$ and the leading terms in $T$ of $\bar{\beta}$, in the one-loop approximation, are given by

$$\bar{\beta} = -\frac{3\lambda^2}{8\pi^2}\left[\frac{1}{2}\,\pi\, e^t - \frac{1}{2}\, t\right] \quad . \tag{11}$$

Solving (10), with $\bar{\beta}$ given by (11), yields

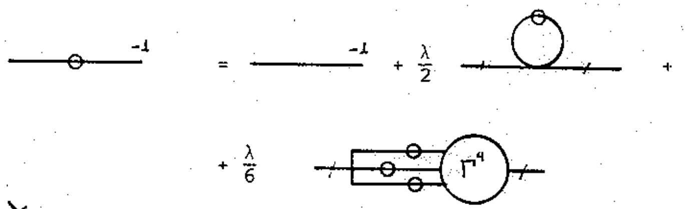
$$\lambda(T) = \frac{\lambda}{1 - \frac{3}{16\pi^2}\,\ell n\left(\frac{T}{m}\right) + \frac{3}{16\pi}\,\frac{T}{m}} \tag{12}$$

As it is well known, the use of Renormalization Group Techniques is an improvemente with respect to the lowest order perturbation theory. This can be easily understood in our case since the result (12) is just the high temperature limit associated to the sum of the series

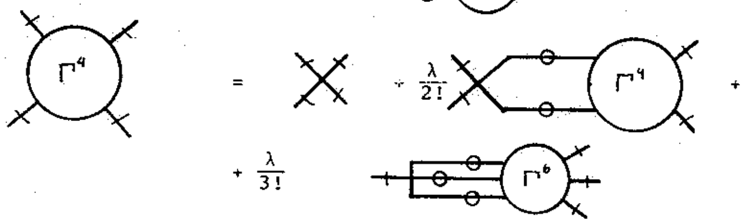
$\lambda(T) = \lambda +$ 

An interesting feature of (12), as has been pointed out by Babu Joseph et al.[4], is that the theory seems to exhibit asymptotic freedom at high temperature - that is, the theory at high temperatures has a different behaviour (with respect to asymptotic freedom) from that at high energies.

Since the above result is quite surprising and in view of the fact that perturbation theory breaks down at high temperatures we would like to use a different, non-perturbative scheme to compute the running coupling constant. We will compute the running coupling constant by finding an approximate solution to the Dyson-Schwinger equations[7]. The Dyson-Schwinger equations for the 2-point and 4-point 1PI Green's functions are[8]:

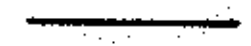


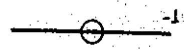
(13)



(14)



where  represents the full propagator, while  stands for the free one. In the high temperature limit $(T >> m)$ we can find a self-consistent solution to equations (13) and (14) which is given by*

 $= \vec{k}^2 + \omega_n^2 + m^2(T)$ (15)

$$\lambda(T) = -\Gamma^{(4)}(P_i = 0, T) = \frac{\lambda}{1 + \frac{3\lambda}{8\pi^2}\left[\frac{\pi T}{2m(T)} + \frac{1}{2}\ell n\left(\frac{m(T)}{4\pi T}\right)\right]} \qquad (16)$$

where $\omega_n = 2\pi n/\beta$, n = interger, and $m^2(T)$ is given by (8).

As we can see from (16), the running coupling constant obtained through the solution to the Dyson-Schwinger equations differs from the one obtained from the renormalization group equation (12). The solution to the Dyson-Schwinger equations represents an improvement of the renormalization group results obtained with $\overline{\beta}$ given by (11), for (16) reduces to (12) for $\lambda T^2 << m^2$, i.e., for $T \approx m\lambda^{-\varepsilon}$ with $0 < \varepsilon < \frac{1}{2}$.

The failure of the renormalization group equations at very high temperatures $(T \approx m\lambda^{-\varepsilon}, \varepsilon > \frac{1}{2})$ is due to the fact that $\overline{\beta}$ has to be evaluated perturbatively and perturbation theory breaks down in the limit of high temperatures[9]. In fact, for very high temperatures we have to add the leading contribution in powers of T of the following diagrams beside the ones in (12)

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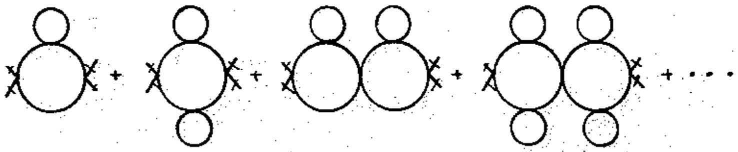
*$\Gamma^6$ is of the order $\lambda^3$, thus negligible.



The sum of these contributions will lead us to the result (16). This is an evidence that (16) is an improvement of the renormalization group result (12).

It is possible to get the result (16) by using an improved expression for $\bar{\beta}$. In order to achieve this we have to evaluate $\bar{\beta}$ using the expression (15) as the propagator which leads us to

$$\frac{d\lambda}{d\tilde{t}} = -\frac{3\lambda^2}{8\pi^2}\left[\frac{1}{2}\pi\, e^{\tilde{t}} - \frac{1}{2}\tilde{t}\right] \qquad (17)$$

where $$\tilde{t} = \ell n\left[\frac{T}{m(T)}\right] . \qquad (18)$$

It is very easy to verify that (16) is a solution to (17). Therefore we can see that, as a result of the breakdown of the perturbative expansion at very high temperatures, the renormalization group equation lead to the right answer only if we evaluate $\bar{\beta}$ carefully.

Now we are prepared to analyze whether or not this model exhibits asymptotic freedom at high temperatures. From (16) we obtain

$$\lim_{T\to\infty} \lambda(T) = \frac{\lambda(0)}{1 + \frac{3\lambda(0)}{16\pi^2}\left\{\ell n\left[\sqrt{\frac{\lambda(0)}{24}}\right] + \pi\sqrt{\frac{24}{\lambda(0)}}\right\}} \qquad (19)$$



where $\lambda(0)$ is the zero temperature value of $\lambda(T)$. It is clear from (19) that in the high temperature limit ($\lambda T^2 \gg m^2$) the coupling constant is essentially independent of the temperature and nonvanishing.

A point which deserves to be further explored is the analysis of the high temperature behaviour of a model which exhibits asymptotic freedom at zero temperature. For the high temperature behaviour of a diagram comes from its infrared it is not trivial to see that the theory will exhibit asymptotic freedom at high temperatures.

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