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A MODEL OF CONFINEMENT IN 2+1 DIMENSIONAL QCD

by

J. Frenkel and A.C. Silva Fo.
Instituto de Física, Universidade de São Paulo

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Instituto de Física, Universidade de São Paulo
C.P. 20516, 01498 São Paulo, SP, Brasil

ABSTRACT

We discuss a dielectric model of QCD in 2-space dimensions which yields confinement of two opposite color charges via a static linear potential. We study analytically and numerically the non-leading contributions to the asymptotic potential as well as the structure of the confinement domain. For large separations of the color charges, we find a behavior which contrasts with the usual string-like picture.

I. INTRODUCTION

In the past years there have been many investigations of classical models of confinement [1-3] which represent approximations of the quantum chromodynamics theory. This approach gives an intuitive picture of QCD vacuum as a dielectric medium, arising in consequence of the quantum fluctuations of Yang-Mills fields [4-6], the property of which gives rise to confinement. In an important serie of papers Adler and Piran [7] have studied many aspects of this problem and developed general numerical methods for its investigation. On the other hand Lehmann and Wu [8] have devised an interesting perturbative treatment for the analytical investigation of the structure of classical models of confinement.

In this work we study, with the help of analytical and numerical methods motivated by these authors, a dielectric model of confinement in 2+1 dimensional QCD. The situation in this case is comparatively simpler than in 3+1 dimensional space-time, due to the fact that the quantum corrections of Yang-Mills fields to the effective coupling constant become very small at short distances. Consequently, by considering infinitely massive sources when quark-antiquark pairs are suppressed, we see that in this case the dielectric vacuum at short distances behaves like in classical electrodynamics where $\varepsilon=1$. In general, the effective QCD Euler-Lagrange equations reduce, in the quasi-abelian approximation for the quark color charges, to those of non-linear electrodynamics with a field dependent dielectric parameter $\varepsilon(E)$ [7]. This approximation is motivated by the fact that, in the infinite quark mass limit, a quark charge cannot be screened by the color octet of the gluon fields. Thus from a color charge in the 3 representation a net color flux emerges which must end on the charge in the $\bar{3}$ representation. The quasi-abelian approximation models this flux as a conserved abelian flux.

In order to see the self-consistency of this approximation, consider the effective static equations of two dimensional QCD which are given by:

$$(\mathcal{D}_j \, \varepsilon E_j)^a = \rho^a \tag{1.a}$$

where a denotes color indices, j represents the spatial coordinates x , y and $\mathcal{D}$ stands for the covariant derivative in terms of which the color field E_j^a is represented as:

$$E_j^a = -(\mathcal{D}_j A_0)^a = -(\partial_j A_0^a + g f^{abc} A_j^b A_0^c) \tag{1.b}$$

with f^{abc} denoting the SU(N) group structure constants. Similarly, the equations determining the magnetic field B^a where:

$$B^a = \partial_x A_y^a - \partial_y A_x^a + g f^{abc} A_x^b A_y^c \tag{2.a}$$

are given by:

$$(\mathcal{D}_j \, \varepsilon B)^a = g \eta_{jj'} f^{abc} A_0^b (\varepsilon E_{j'}^c) \tag{2.b}$$

where η is a two dimensional antisymmetric tensor with $\eta_{xy} = 1$. With the help of the identity:

$$([\mathcal{D}_x, \mathcal{D}_y] F)^a = g f^{abc} B^b F^c \tag{3}$$

applied to equation (2), it is easy to show making use of (1), that in order to obtain consistent static solutions the following condition must be obeyed:

$$f^{abc} A_0^b \rho^c = 0 \quad . \tag{4}$$

This constraint is satisfied if $A_0^b = V^b A_0$ and $\rho^c = V^c \rho$, where $\hat{V}$ is a fixed unit vector in the color space. Furthermore if we make the quasi-abelian ansatz $A_j^a = V^a A_j$ so that $E_j^a = V^a E_j$ and $B^a = V^a B$, equations (1) and (2) reduce respectively to those of non-linear static two dimensional electromagnetism, with a field dependent dielectric parameter ε :

$$\partial_x(\varepsilon E_x) + \partial_y(\varepsilon E_y) = \vec{\nabla} \cdot (\varepsilon \vec{E}) = \rho \tag{5.a}$$

$$\partial_x(\varepsilon B) = \partial_y(\varepsilon B) = 0 \tag{5.b}$$

to which we must also add the equation describing the irrotational nature of the electric field:

$$\partial_x E_y - \partial_y E_x = 0 \quad . \tag{5.c}$$

Equation (5.b) states that εB must be a constant. Since we are interested in configurations with minimal energy, which are obtained for vanishing magnetic fields [7] we will have $B = 0$.

We must now indicate the manner in which $\varepsilon(E)$ depends on the electric field. As it will be justified in the next section, we take this relation to be given by:

$$\varepsilon(E) = 1 - \frac{E_0}{E} \quad , \quad \text{for} \quad E \geq E_0 \qquad (6)$$

where $E_0 > 0$ denotes an arbitrary, fixed electric field. Despite the simplicity of this form, we shall see that no exact solutions are available due to the intrinsic non-linear nature of the problem. However many interesting features can be revealed in this case, such as the transverse width of the confinement domain as well as the behaviour of subdominant terms in the static potential at large distances. In section II we present the model and formulate an analytic approach which gives its asymptotic properties in the case of two opposite static charges separated by a large distance R. In section III we evaluate the first order corrections arising in a perturbative treatment and discuss some general features characterizing the boundary which delimits the confinement domain. Finally in section IV we describe a numerical analysis of the problem and find an excellent agreement with the analytic results. These results indicate a transverse width of the confinement domain of order $R^{2/3}$ as well as non-leading contributions to the linear potential which are proportional to $R^{1/3}$. This behaviour is in contrast with the usual picture which assumes a string-like form for the flux tube connecting the color charges at large distances.

II. THE MODEL AND ITS ASYMPTOTIC PROPERTIES

In order to justify the choice (6) for the dielectric parameter consider the constitutive relation for the displacement field $\vec{D}$ given by $\vec{D} = \varepsilon(E)\vec{E}$. By inverting it we can express

the electric field as a function of D: $E = E_0 f(D)$. When the following conditions are satisfied:

$$f = 1 \quad \text{at} \quad D = 0 \qquad (7.a)$$

$$f \geq 1 \quad \text{for} \quad E \geq E_0 \qquad (7.b)$$

it was shown by t'Hooft [1] that the electrostatic energy of two point charges increases linearly for large separations. In this case $E = E_0$ determines a boundary of the confinement domain, outside which the energy density vanishes and consequently $D=0$. The class of models we are concerned with [7,8] make the physically reasonable assumption that the D fields are continuous across the boundary so that D increases from zero towards the interior of the confinement region. Since the 2+1 dimensional QCD leads to the confinement of the color charges by a linear potential, as shown by Feynman [9], it is reasonable to assume that the conditions given in (7) are satisfied. Due to the continuity of the D field, we can make a Taylor expansion in the interior of the domain in the vicinity of the boundary as follows:

$$f(D) = 1 + cD + O(D^2) \quad , \quad \text{for} \quad D \ll E_0 \,. \qquad (8.a)$$

On the other hand, for large values of the electric fields which occur at small distances in the neighbourhood of the color charges, as we have mentioned we have $\varepsilon \cong 1$, i.e $D \cong E$. Using the relation $E = E_0 f(D)$, this condition can be written as follows:

$$f(D) \cong \frac{1}{E_0} D \quad , \quad \text{for} \quad D \gg E_0 \quad . \qquad (8.b)$$

Of course the exact form of $f(D)$ is unknown until we solve completely the QCD theory. For this reason we choose the simplest analytic expression consistent with the boundary conditions (7,8) and the continuity of the D field, namely:

$$f(D) = 1 + \frac{1}{E_0} D \qquad (9)$$

which then leads to the form indicated in equation (6). [Alternative bag models have discontinuous dielectric constant and D fields across the boundary $E = E_0$. A simple model of this type with $\varepsilon = \text{constant}$ for $E > E_0$ and $\varepsilon = 0$ for $E < E_0$ was investigated [10] and shown to be exactly soluble for two space dimensions].

A solution of the non-linear differential equation (5) with the dielectric parameter $\varepsilon(E)$ given by (6), depends in a crucial way on the boundary conditions which must be satisfied by the fields. In such equations the boundary conditions cannot be imposed a priori, in general, but depend on the solution. A convenient way of handling this situation is via the flux function introduced by Adler. In this method one expresses E as a function of the electric flux ϕ, on which the boundary conditions can be expressed in a convenient way. To see this, note that outside the sources, which we take to be two opposite point charges separated by a large distance $R = 2a$, equation (5.a) can be written as

$$\vec{\nabla} \cdot (\varepsilon \vec{E}) = \vec{\nabla} \cdot \vec{D} = 0 \qquad (10.a)$$

which is satisfied if $\vec{D}$ is expressed as the rotational of a function ϕ:

$$D_x = \frac{\partial \phi}{\partial y} \quad ; \quad D_y = -\frac{\partial \phi}{\partial x} \quad . \qquad (10.b)$$

The physical interpretation of $\phi = \int \vec{D} \cdot d\vec{c}$ is that of the electric flux through the curve C, which intersects the charge axis at a point $x > a$, as shown in figure 1.

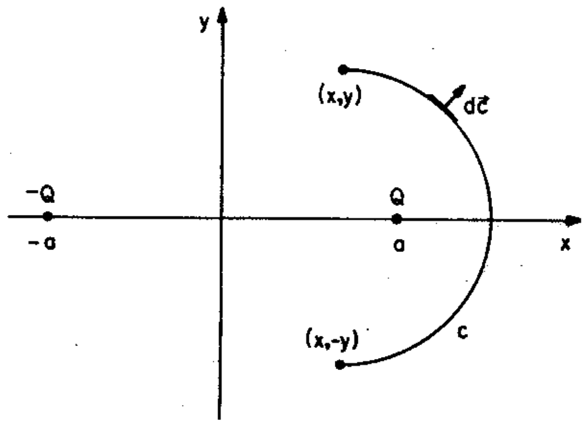

Figure 1 - The curve C used in the evaluation of the flux, bounded by the symmetrical points (x,y) and (x,-y).

The boundary conditions on the flux function $\phi(x,y)$ are:

$$\phi(x,0) = \begin{cases} Q & \text{if} \quad |x| < a \\ 0 & \text{if} \quad |x| > a \end{cases} \qquad (11)$$

together with the requirement that ϕ vanishes when $x^2+y^2 \to \infty$, which ensures that no additional sources are present at spatial infinity.

By expressing E in function of ϕ with the help of

(10.b), equation (5.c) becomes:

$$\nabla^2\phi + E_0\vec{\nabla}\cdot\left(\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\right) = 0 \qquad (12)$$

In general this non-linear equation cannot be solved in closed form. When $D = |\vec{\nabla}\phi| >> E_0$, which is the situation near the sources, (12) reduces to the linear form of the electrostatic theory, which is characterized by $\varepsilon=1$ and hence $E_0=0$. In this case the solution satisfying the boundary condition (11) is given by:

$$\phi(x,y) = \frac{Q}{\pi}\left[\tan^{-1}\left(\frac{y}{x-a}\right) - \tan^{-1}\left(\frac{y}{x+a}\right)\right] . \qquad (13)$$

Apart from this case we must resort to perturbative methods in order to solve (12). To this end it is convenient to rescale ϕ, x and y in terms of dimensionless parameters, $\phi \to Q\phi$; $(x,y) \to \frac{Q}{E_0}(x,y)$ which are to be understood in what follows. In other words, all quantities are to be expressed in units of charge Q and length $\frac{Q}{E_0}$. Expanding (12) we find in this way the equation:

$$\left\{\left[\phi_x^2+\phi_y^2\right]^{3/2} + \phi_y^2\right\}\phi_{xx} + \left\{\left[\phi_x^2+\phi_y^2\right]^{3/2} + \phi_x^2\right\}\phi_{yy}$$

$$- 2\phi_x\phi_y\phi_{xy} = 0 \qquad (14.a)$$

where

$$\phi_x = \frac{\partial\phi}{\partial x} \; ; \; \phi_{xx} = \frac{\partial^2\phi}{\partial x^2} \; ; \; \phi_{xy} = \frac{\partial^2\phi}{\partial x\partial y} \quad \text{e.t.c. .} \qquad (14.b)$$

In order to get an orientation toward the solution of this

equation we rescale the coordinates x and y respectively by a and a^α. Since $\alpha=1$ for ordinary electrostatic, we expect $\alpha<1$ in a confining model. In this case the dominant order of magnitude of the three sets of terms in (14.a) are respectively $a^{-2-2\alpha}$, $a^{-2-2\alpha}+a^{-5\alpha}$, $a^{-2-2\alpha}$. These orders are comparable if $\alpha=2/3$. With this motivation we introduce the variables

$$x = as \quad ; \quad y = a^{2/3}\,t \qquad (15.a)$$

in terms of which equation (14) becomes:

$$\left\{\left[a^{-2/3}\phi_s^2+\phi_t^2\right]^{3/2} a^{-2/3} + \phi_t^2\right\}\phi_{ss} + \left\{\left[a^{-2/3}\phi_s^2+\phi_t^2\right]^{3/2} + \phi_s^2\right\}\phi_{tt}$$

$$- 2\phi_s\phi_t\phi_{st} = 0 \quad . \qquad (15.b)$$

This form shows that the relevant expansion parameter is $a^{-2/3}$. Then, provided the condition $a^{-2/3}\phi_s^2 << \phi_t^2$ is satisfied, (15.b) reduces to the equation

$$\tilde{\phi}_t^2\tilde{\phi}_{ss} + \left[\tilde{\phi}_t^3+\tilde{\phi}_s^2\right]\tilde{\phi}_{tt} - 2\tilde{\phi}_s\tilde{\phi}_t\tilde{\phi}_{st} = 0 \qquad (16.a)$$

determining the asymptotic form of the flux function which we denote by $\tilde{\phi}$. A solution to this equation can be obtained by expressing t as a function of $\tilde{\phi}$ and s. The resulting partial differential equation can then be solved by the method of separation of variables. Using the boundary condition (11) we obtain for $\tilde{\phi}$ result:

$$\tilde{\phi}(s,t) = 1 - \frac{2^{-2/3}|t|}{(1-|s|)^{2/3}} + \frac{1}{27}\frac{|t|^3}{(1-|s|)^2} \quad . \qquad (16.b)$$

In terms of the original variables x,y given in (15.a), we can express $\tilde{\phi}$ in the form

$$\tilde{\phi}(x,y) = \left[1 - \frac{2^{1/3}}{3} \frac{|y|}{(a-|x|)^{2/3}}\right]^2 \cdot \left[1 + \frac{2^{-2/3}}{3} \frac{|y|}{(a-|x|)^{2/3}}\right] \qquad (17)$$

From this form we see that the characteristic curve along which $\tilde{\phi}$ together with its first derivatives $\tilde{\phi}_x$ and $\tilde{\phi}_y$ vanish is given by

$$|\tilde{y}^b(x)| = 3.2^{-1/3} (a-|x|)^{2/3} \quad . \qquad (18)$$

This curve delimits the domain confining the asymptotic flux function $\tilde{\phi}$, which positions the charges on the characteristic boundary.

We now proceed to calculate the total electrostatic energy:

$$V(R) = \int dx\, dy \int_0^D E(D')\, dD' = \int dx\, dy\, (D + \tfrac{1}{2} D^2) \quad . \qquad (19)$$

Using the symmetry of our problem we obtain:

$$V(R) = 4 \int_0^{R/2} dx \int_0^{|\tilde{y}^b(x)|} dy\, (D + \tfrac{1}{2} D^2) \qquad (20.a)$$

where $\tilde{y}^b(x)$ is given by (18) and, up to corrections of order $R^{-2/3}$:

$$D = |\vec{\nabla}\phi| \cong \tilde{\phi}_y \left[1 + \frac{1}{2} (\tilde{\phi}_x/\tilde{\phi}_y)^2\right] \qquad (20.b)$$

with $\tilde{\phi}$ denoting the asymptotic flux function (17). Performing the integrations over the energy density $H = D + \frac{1}{2} D^2$ we find:

$$V(R) = 2R\left[1 + \frac{12}{5} R^{-2/3} + \ldots\right] \quad . \qquad (21)$$

This expression shows that, in addition to the dominant linear contributions to the static potential, there are subdominant terms behaving like $R^{1/3}$ for large separations of the charges.

III. PERTURBATIVE CORRECTIONS AND BOUNDARY BEHAVIOR

As we have seen in our case the relevant expansion parameter is $a^{-2/3}$ so that we can write the solution to the flux function as a perturbative serie:

$$\phi = \phi^0 + a^{-2/3} \phi^1 + \ldots \qquad (22)$$

where ϕ^0 is given by the asymptotic form (17), up to corrections of order $a^{-2/3}$.

Substituting this expression into (15) we find, to the required accuracy, the following equation:

$$(\phi_t^0)^2 \phi_{ss}^0 + \left[(\phi_t^0)^3 + (\phi_s^0)^2\right] \phi_{tt}^0 - 2\phi_s^0 \phi_t^0 \phi_{st}^0 + a^{-2/3} E = 0 \qquad (23.a)$$

where E is given by:

$$E = (\phi_t^0)^2 \phi_{ss}^1 + \left[(\phi_t^0)^3 + (\phi_s^0)^2\right] \phi_{tt}^1 - 2\phi_s^0 \phi_t^0 \phi_{st}^1 + 2\left[\phi_s^0 \phi_{tt}^0 - \phi_t^0 \phi_{st}^0\right] \phi_s^1$$
$$+ 2\left[\phi_{ss}^0 \phi_t^0 - \phi_s^0 \phi_{st}^0 + \frac{3}{2} \phi_{tt}^0 (\phi_t^0)^2\right] \phi_t^1 + (\phi_t^1)^3 \phi_{ss}^0 + \frac{3}{2} (\phi_s^0)^2 \phi_t^0 \phi_{tt}^0 \quad . \qquad (23.b)$$

This represents a fairly complicated expression which can however be simplified by choosing appropriate variables. We recall that in non-linear equations the solution and boundary conditions are strongly interdependent, so this choice must be made in a way which allows these conditions to be more conveniently expressed. To this end let us expand perturbatively the form giving the characteristic boundary curve. With the help of equations (18) and (15.a) we have:

$$|t^b(s)| = 3.2^{-1/3}(1-|s|)^{2/3} + a^{-2/3}\,\tilde{f}(s) + \ldots$$

$$\equiv \tilde{t}^b(s) + a^{-2/3}\,\tilde{f}(s) + \ldots \quad . \qquad (24.a)$$

Then, motivated by the work in reference [8] we introduce a new variable u:

$$u = t/t^b(s) \quad ; \quad t = t^b(s)u \qquad (24.b)$$

which vanishes at $t=0$ and equals unity on the boundary. After expressing t in terms of the variable u and choosing $\phi^0 = (u-1)^2(\frac{u}{2}+1)$, we find that equation (23) reduces to:

$$(u^2-1)u\left[-f_{ss} + \frac{4}{3}(1-|s|)^{-1} f_s + \frac{2}{3}(1-|s|)^{-2} f\right] +$$

$$+ \frac{5}{3}\frac{1}{\tilde{t}^b}(u^2-1)u(3u^2-1)(1-|s|)^{-2} +$$

$$+ \frac{2}{3}\left[\psi^1_{ss} + \frac{3}{2}\frac{1}{(\tilde{t}^b)^3}(u^2-1)\,\psi^1_{uu} + 2\frac{\tilde{t}^b_s}{\tilde{t}^b}\,\psi^1_s\right] = 0 \qquad (25)$$

where in the notation of equation (24.a) we have $f = \tilde{f}/\tilde{t}^b$. Since the form chosen for ϕ^0 is such that it equals unity at $u=0$ while together with its first derivative vanishes on the boundary $u=1$, condition (11) together with the requirement that the boundary should be a characteristic curve requires:

$$\phi^1(u=0) = 0 \quad \text{and} \quad \phi^1(u=1) = \phi^1_u(u=1) = 0 \quad . \qquad (26.a)$$

Then from equation (25) we see that ϕ^1 must have the form:

$$\phi^1(s,u) = u(u^2-1)^2\,h(s) \quad . \qquad (26.b)$$

Substituting this expression into (25) yields two ordinary differential equations which determine the behaviour of the functions $f(s)$ and $h(s)$:

$$h_{ss} - \frac{4}{3}(1-|s|)^{-1} h_s - \frac{20}{9}(1-|s|)^{-2} h + \frac{15}{2\tilde{t}^b}(1-|s|)^{-2} = 0 \qquad (27.a)$$

$$\frac{3}{2}\left[-f_{ss} + \frac{4}{3}(1-|s|)^{-1} f_s + \frac{2}{3}(1-|s|)^{-2} f\right] - \frac{8}{9}(1-|s|)^{-2} h + \frac{5}{\tilde{t}^b}(1-|s|)^{-2} = 0 . \qquad (27.b)$$

When solving these equations we must realize that these functions must yield, from (22) and (24), small perturbative corrections to the asymptotic flux function and its boundary. With this constraint we find for h and f the following expressions:

$$h(s) = (1-|s|)^{-2/3}\left[C_0(1-|s|)^2 - \frac{5}{2^{5/3}}\right] \qquad (28.a)$$

$$f(s) = (1-|s|)^{-2/3}\left[C_1(1-|s|)^{4/3} - \frac{8C_0}{21}(1-|s|)^2 + \frac{5/3}{2^{2/3}}\right] \quad . \qquad (28.b)$$

Using these results we obtain for the flux function including first order corrections, the expression:

$$\phi \cong (u-1)^2 \left(\frac{u}{2}+1\right) + a^{-2/3}\, u(u^2-1)^2 \left[C_0 (1-|s|)^{4/3} - \frac{5}{2^{5/3}} (1-|s|)^{-2/3}\right] \qquad (29)$$

with the boundary t^b given by:

$$|t^b| \cong \frac{3}{2^{1/3}} \left\{(1-|s|)^{2/3} + a^{-2/3} \left[C_1 (1-|s|)^{4/3} - \frac{8C_0}{21} (1-|s|)^2 + \frac{5/3}{2^{1/3}}\right]\right\}. \qquad (30)$$

The determination of C_0 and C_1 would require information about the solution near the charges situated at $|s| = 1$, where D is large. However since a necessary condition for the applicability of the perturbative method is that D is small, the above solutions are not valid near the charges. This can be seen also from the remarks following equation (15), where it was assumed that $a^{-2/3}\phi_s^2 << \phi_t^2$. This condition does not hold in the region $|s| \approx 1$, as can be seen from the asymptotic expression (16.b).

In order to gain further insight of the behaviour of the boundary we will use an alternative approach [7]. This is obtained by putting equation (12) into the standard form for a second order differential equation and analysing the structure of its characteristics. Defining the inward directed unit normal $\hat{n}$ and the corresponding normal derivative ∂_n:

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} \quad ; \quad \partial_n = \hat{n} \cdot \partial \quad . \qquad (31.a)$$

we obtain through a straightforward calculation that (12) is equivalent to:

$$(\partial_x^2 + \partial_y^2 - \partial_n^2)\phi + \frac{|\vec{\nabla}\phi|}{1+|\vec{\nabla}\phi|}\, \partial_n^2 \phi = 0 \quad . \qquad (31.b)$$

An approximate solution of this equation near the boundary where $\phi=0$ can be obtained through the introduction of tangencial and normal coordinates denoted respectively by ℓ and n, as shown in figure 2.

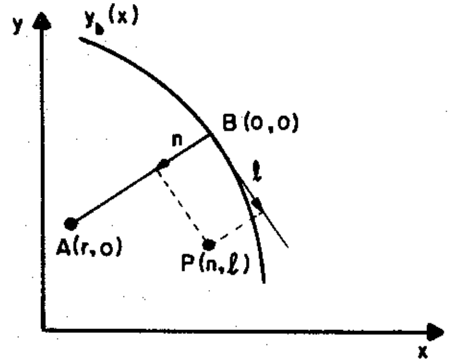

Figure 2 - Representation of normal and tangencial coordinates near the boundary $y_b(x)$.

Then equation (31) reduces in this region to:

$$\partial_\ell^2\phi + |\vec{\nabla}\phi|\, \partial_n^2\phi = 0 \quad . \qquad (32)$$

When the radius of curvature at B is r, then to a good approximation the boundary in the vicinity of B is described by a circle characterized by $n = \ell^2/2r$. Since ϕ vanishes on the boundary we see that to the needed accuracy in this region ϕ has the form:

$$\phi = F\left(n - \frac{1}{2}\frac{\ell^2}{r}\right) \quad ; \quad F(0) = 0 \quad . \qquad (33)$$

Substituting (33) into (32) determines ϕ to be given in leading order by:

$$\phi \cong \frac{1}{2r}\left[n - \frac{1}{2}\frac{\ell^2}{r}\right]^2 \quad . \qquad (34)$$

Because ϕ decreases towards the boundary for increasing values of ℓ, we see that one must have $r > 0$ which implies that the boundary is everywhere convex.

IV. NUMERICAL ANALYSIS

In order to formulate the continuum problem of equation (12) which in our units can be written as:

$$\vec{\nabla} \cdot \left\{ \left[\frac{1}{|\vec{\nabla}\phi|} + 1 \right] \vec{\nabla}\phi \right\} \equiv \vec{\nabla} \cdot \{h\vec{\nabla}\phi\} = 0 \qquad (35)$$

we must approximate it by a similar problem defined on a discrete computational lattice. We introduce it, following the work in reference [7], by replacing the continuous variables x and y by discrete ones:

$$x_i = i\Delta \quad ; \quad y_j = j\Delta \qquad i,j = \ldots -1,0,1 \ldots \qquad (36.a)$$

and the corresponding half-node lattice:

$$x_i + \frac{1}{2} = (i + \frac{1}{2})\Delta \quad ; \quad y_j = (j + \frac{1}{2})\Delta \qquad i,j = \ldots -1,0,1 \ldots \qquad (36.b)$$

In discretizing our model we put the charge coordinates $x = \pm a$ on a node of the computational lattice:

$$a = n\Delta \qquad (37.a)$$

and enforce the step function boundary condition (11) by requiring:

$$\begin{aligned} \phi_{i,0} &= 1 \qquad 0 < |i| < n \\ \phi_{i,0} &= \frac{1}{2} \qquad |i| = n \qquad (37.b) \\ \phi_{i,0} &= 0 \qquad n < |i| \leq n_i \end{aligned}$$

where $\phi_{i,j}$ denotes the values of the flux function on the lattice. Because the solution of ϕ is confined within a finite boundary, the numerical solution is independent of x_{max}, y_{max} provided that these are large enough for the boundary to lie entirely within the computational mesh.

Let $h_{i+\frac{1}{2},j+\frac{1}{2}}$ denote the values of h on the half-node lattice. Working to second-order accuracy Δ^2 we can then replace the differential operators by finite differences. In terms of this discretization, the iteration consists of alternating a complete sweep of the node lattice with a complete sweep of the half node lattice. In the sweep of the node lattice $\phi^n_{i,j}$ is updated by the iteration $\phi^n_{i,j} \rightarrow \phi^{n+1}_{i,j}$ where:

$$\begin{aligned} \phi^{n+1}_{i,j} = \frac{1}{2}\Big[h^n_{i+\frac{1}{2},j+\frac{1}{2}} + h^n_{i+\frac{1}{2},j-\frac{1}{2}} + h^n_{i-\frac{1}{2},j+\frac{1}{2}} + h^n_{i-\frac{1}{2},j-\frac{1}{2}}\Big]^{-1} \\ \times \Big[\Big(h^n_{i+\frac{1}{2},j+\frac{1}{2}} + h^n_{i+\frac{1}{2},j-\frac{1}{2}}\Big)\phi^n_{i+1,j} + \Big(h^n_{i-\frac{1}{2},j+\frac{1}{2}} + h^n_{i-\frac{1}{2},j-\frac{1}{2}}\Big)\phi^{n+1}_{i-1,j} \\ + \Big(h^n_{i+\frac{1}{2},j+\frac{1}{2}} + h^n_{i-\frac{1}{2},j+\frac{1}{2}}\Big)\phi^n_{i,j+1} + \Big(h^n_{i+\frac{1}{2},j-\frac{1}{2}} + h^n_{i-\frac{1}{2},j-\frac{1}{2}}\Big)\phi^{n+1}_{i,j-1}\Big] \end{aligned} \qquad (38)$$

while in the subsequent sweep of the half-node lattice, $h^n_{i+\frac{1}{2},j+\frac{1}{2}}$ is updated by the replacement $h^n_{i+\frac{1}{2},j+\frac{1}{2}} \rightarrow h^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}$ where:

$$h^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = \left[\frac{1}{|\vec{\nabla}\phi|^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}} + 1\right] \qquad (39.a)$$

$$|\vec{\nabla}\phi|^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = \left\{\frac{1}{2}\left[\frac{\phi^{n+1}_{i+1,j} - \phi^{n+1}_{i,j}}{\Delta}\right]^2 + \frac{1}{2}\left[\frac{\phi^{n+1}_{i+1,j+1} - \phi^{n+1}_{i,j+1}}{\Delta}\right]^2\right.$$

$$\left. + \frac{1}{2}\left[\frac{\phi^{n+1}_{i,j+1} - \phi^{n+1}_{i,j}}{\Delta}\right]^2 + \frac{1}{2}\left[\frac{\phi^{n+1}_{i+1,j+1} - \phi^{n+1}_{i+1,j}}{\Delta}\right]^2\right\}^{\frac{1}{2}} . \qquad (39.b)$$

Sample results on a 240×240 mesh computed for the parameters $Q=1=E_0$ for various separations $R = 2a$ of the charges are presented in figure 3. This shows the flux function ϕ and the energy density H plotted vertically over an horizontal plane through the charge axis. From the figure we can clearly see the boundary condition on ϕ at $y=0$ as well as the Coulomb self-energy peaks associated with H. We also observe that ϕ and H are confined within a finite domain bounded by a characteristic curve. The shape of ϕ agrees well with the one obtained analytically in equations (17-18), reflecting the fact that the distance between the charges and the boundary is of order 1. This behaviour was actually assumed in the derivation of the subdominant terms in the analytic expression of the potential (21). Integrating numerically H over the confinement domain, we obtain the following fit for the potential:

$$V(R) = 2.01\, R\,(1 + 2.21\, R^{-\frac{2}{3}} + \dots) \qquad (40)$$

which agrees within one percent with the analytical result given in equation (21). The agreement is excellent given that,

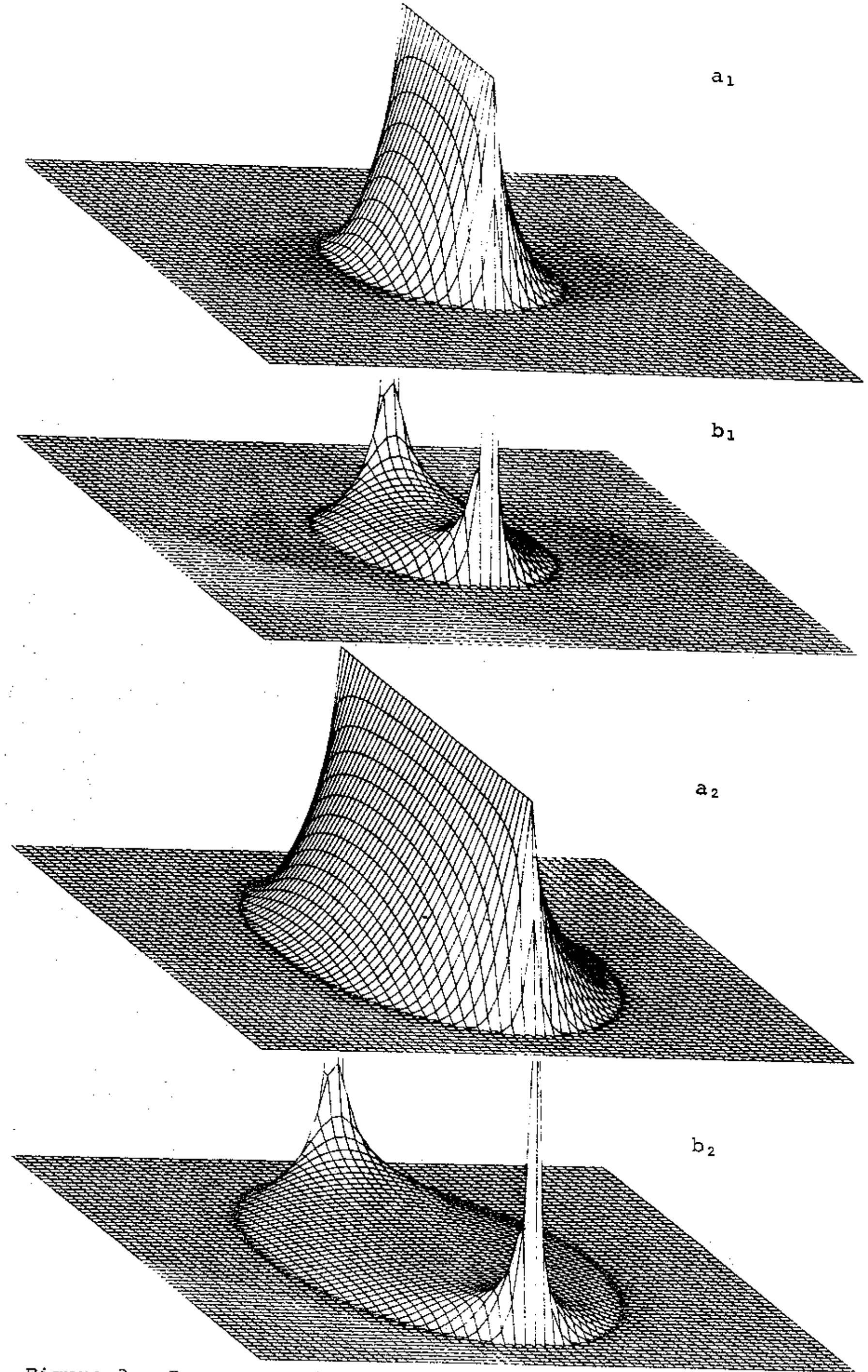

Figure 3 - Representation of the flux function $\phi\,(a_1,a_2)$ and energy density $H\,(b_1,b_2)$ for charge separations $R_1 = 48$ and $R_2 = 96$, respectively.

due to the limitations of the available computer facilities, the maximum range of R was limited to moderately large values of order of 100.

Finally we note that the transversal width of the confinement domain grows like $R^{2/3}$ for large values of R. A similar feature occurs in 3-dimensional models [7,8] when the width behaves asymptotically like $R^{1/2}$ if condition (8a) is satisfied. We can understand this behavior by considering two opposite charges $\pm Q$ separated by a large distance R in a $N \geq 2$ dimensional space. If we denote by L the transversal width of the confinement domain in the region equidistant from the charges, we obtain that the total flux Q is proportional to σL^{N-1}, where σ denotes the mean flux density. We expect qualitatively σ to be inversely proportional to some power of R: $\sigma \sim 1/R^p$. Hence it follows from these considerations that the transversal width behaves for large values of R like $L \sim R^{p/(N-1)}$. The results obtained in the class of models studied above indicate that p is a parameter of order unity. This is very different from the traditional flux tube picture where, essentially, one assumes that p is a vanishing quantity for large separations between the color charges.

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