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LOGARITHMIC CORRECTIONS TO THE UNCERTAINTY PRINCIPLE AND INFINITUDE OF THE NUMBER OF BOUND STATES OF N-PARTICLE SYSTEMS

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ABSTRACT

We show that critical long distance behaviour for a two-body potential, defining the finiteness or infinitude of the number of negative eigenvalues of Schrödinger operators in ν-dimensions, are given by $v_k(r) = -\left(\frac{\nu-2}{2r}\right)^2 - \frac{1}{(2r\,\ell n\, r)^2} + \cdots - \frac{1}{(2r\,\ell n\, r.\ell n\,\ell n\, r \ldots \ell n_{(k)} r)^2}$ where $k=0,1,\ldots$ for $\nu\neq 2$ and $k=1,2,\ldots$ if $\nu=2$. This result is a consequence of logarithmic corrections to an inequality known as Uncertainty Principle.

If the continuum threshold in the N-body problem is defined by a two-cluster break up our results generate corrections to the existing sufficient conditions for the existence of infinitely many bound states.

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I. INTRODUCTION

It is well known that the finiteness or infinitude of the number of bound-states of negative energy of a Schrödinger operator $[-\Delta+V]$ is controlled by the long-distance behaviour of the potential [1,2,4,11]. For dimension $\nu\neq 2$ a finiteness-infinitude borderline is set by a fall-off $\sim \left(\frac{\nu-2}{2r}\right)^2$ as $r\to\infty$. Not coincidentally, for the quadratic form $(\psi,[-\Delta+V]\psi)$, $\psi\in C_0^\infty(\mathbb{R}^\nu\setminus 0)$ and V being a Kato potential†, the following results hold:

A) "Uncertainty Principle Lemma" [2,3,8,12]

If $V(x) \geq -\left(\frac{\nu-2}{2r}\right)^2$ then

$$(\psi,[-\Delta+V]\psi) \geq 0 \tag{1.1}$$

and

B) If, for $\alpha>1$, $r\geq R_0>0$, $V(x)\leq -\alpha\left(\frac{\nu-2}{2r}\right)^2$ then there exists an infinite sequence $\{\psi_n \in C_0^\infty(\mathbb{R}^\nu\setminus 0)\}_{n\geq 1}$, with disjoint supports, such that

$$(\psi_n,[-\Delta+V]\psi_n) < 0 \quad . \tag{1.2}$$

†Through this paper we will assume that the potential functions satisfy the Kato condition, $V\in L^2(\mathbb{R}^\nu)+L^\infty_\varepsilon(\mathbb{R}^\nu)$ i.e., for any $\varepsilon>0$, there exists a decomposition

$$V = V_{1,\varepsilon} + V_{2,\varepsilon} \quad \text{with} \quad V_{1,\varepsilon}\in L^2(\mathbb{R}^\nu),\ V_{2,\varepsilon}\in L^\infty(\mathbb{R}^\nu)$$

and $\|V_{2,\varepsilon}\|_\infty < \varepsilon$.

This condition will ensure self-adjointness of the relevant Hamiltonians, both for the two-body and the N-body case. [2].

From A) (as proved by Simon [1] for $\nu=3$) it follows that if $V(x) \geq -\left(\frac{\nu-2}{2r}\right)^2$ for $r \geq R_0 > 0$, then $[-\Delta+V]$ has at most finitely many negative eigenvalues. Under the assumptions of B), the "min-max principle" implies the existence of infinitely many eigenstates of negative energy.

For $\nu=2$, however, property A is trivial and property B is false!

The original purpose of our investigation was to determine the critical asymptotic behaviour of the potential for $\nu=2$. The answer is that for $\nu=2$ the critical (in the same sense as above) long distance fall-off is $\sim -\frac{1}{(2r\ln r)^2}$. This follows from appropriately modified versions of A and B above.

Nevertheless, it turns out that the $\nu=2$ result is only the first term of an infinite series of logarithmic corrections for $\nu=1$ and 3 results! This is a consequence of the following chain of facts:

i) Under suitable domain restrictions, the unitary operator T_ν: $L^2(\mathbb{R}_+, r^{\nu-1}dr) \longrightarrow L^2(\mathbb{R}_+, dr)$, $(T_\nu\psi)(r) = r^{\frac{\nu-1}{2}}\psi(r)$ establishes a unitary equivalence between the radial part of the 2-dimensional Laplacean and the critically perturbed radial part of the ν-dimensional Laplacean:

$$T_2\left(-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)T_2^{-1} =$$

$$= T_\nu\left[-\frac{1}{r^{\nu-1}}\frac{d}{dr}r^{\nu-1}\frac{d}{dr} - \left(\frac{\nu-2}{2r}\right)^2\right]T_\nu^{-1} = -\frac{d^2}{dr^2} - \frac{1}{4r^2} \quad (1.3)$$

More generally, if $a: \mathbb{R}_+\setminus N_a \to \mathbb{R}_+$ is C^∞ and $a(r) > 0$ for all $r \in \mathbb{R}_+\setminus N_a$, where N_a is a finite set, then the unitary operator U_a: $L^2(\mathbb{R}_+, a\,dr) \longrightarrow L^2(\mathbb{R}_+, dr)$, given by $(U_a\psi)(r) = a^{1/2}\psi(r)$ transforms the "radial a-Laplacean" as:

$$U_a\left(-\frac{1}{a}\frac{d}{dr}a\frac{d}{dr}\right)U_a^{-1} = \left[-\frac{d^2}{dr^2} - \frac{1}{4}\left(\frac{a'}{a}\right)^2 + \frac{1}{2}\left(\frac{a''}{a}\right)\right] \quad (1.4)$$

when restricted, for instance, to $C_0^\infty(\mathbb{R}_+\setminus N_a)$. (From now on we shall use a prime to denote derivatives with respect to r.)

<u>Remark</u>. Since $\left(-\frac{1}{a}\frac{d}{dr}a\frac{d}{dr}\right)$ is a positive operator, when restricted to $C_0^\infty(\mathbb{R}_+\setminus N_a)$, (1.3) provides a trivial proof of the "Uncertainty Principle Lemma".

ii) For a class of functions $a(r)$ as above, it is possible to find a critical potential v_a for the a-Laplacean. It is given by

$$v_a(r) = -\frac{1}{(2a(r)h(r))^2} \quad (1.5)$$

where h is a monotonic function satisfying

$$h'(r) = \frac{1}{a(r)} \quad (1.6)$$

In fact, denoting by S_a the finite set where a or h are zero, we prove

<u>Lemma 1</u>. If $\psi \in C_0^\infty(\mathbb{R}_+\setminus S_a)$ then

$$\int(\psi')^2 a\,dr \geq \int v_a\psi^2 a\,dr \quad . \quad (1.7)$$

<u>Lemma 2</u>. If $\lim_{r\to\infty} h(r) = \infty$ then, given $\varepsilon > 0$ arbitrary, there exists an infinite family of non-zero functions, with disjoint

supports, $\{\psi_n \in C_0^\infty(\mathbb{R}_+ \setminus S_a)\}_{n\geq 1}$ such that

$$\int (\psi_n')^2 a\,dr < (1+\varepsilon) \int v_a \psi_n^2 a\,dr \; . \tag{1.8}$$

Remarks. (1.7) is a version of an inequality of Hardy [2,3,8,12] known as the "Uncertainty Principle Lemma". Lemma 2 says that the constants appearing in the definition v_a are best possible.

iii) Finally, the whole procedure may be iterated provided we can find $b: \mathbb{R}_+ \setminus N_b \to \mathbb{R}_+$, with the same assumed properties of $a(r)$, such that

$$U_b\left(-\frac{1}{b}\frac{d}{dr} b \frac{d}{dr}\right)U_b^{-1} = U_b\left[-\frac{1}{a}\frac{d}{dr} a \frac{d}{dr} + v_a(r)\right]U_a^{-1} \; . \tag{1.9}$$

Starting with $a=r$ and iterating the whole procedure we obtain the result that the potentials

$$v_k(r) = -\left(\frac{\nu-2}{2r}\right)^2 - \left(\frac{1}{2r\ln r}\right)^2 - \cdots - \left(\frac{1}{2r\ln r \ln_{(2)} r \cdots \ln_{(k)} r}\right)^2 \tag{1.10}$$

for $k \geq 0$ if $\nu \neq 2$ and $k \geq 1$ if $\nu = 2$ are critical, i.e., for some $r \geq R_0 > 0$,

(a) if $V(\vec{x}) > (1+\varepsilon) v_k(r)$ then $[-\Delta+V)$ has finitely many negative eigenvalues or

(b) if $V(x) < v_{k-1}(r) - \dfrac{1+\varepsilon}{(2r\ln r \cdots \ln_{(k)} r)^2}$, for some $\varepsilon > 0$, then $[-\Delta+V]$ has infinitely many negative eigenvalues.

Notation: For $k \geq 2$, $\ln_{(k)} r = \ln \ln_{(k-1)} r$,
and $\ln_{(1)} r = \ln r$.

Our results amount, in fact, to logarithmic corrections to the "Uncertainty Principle", a widely used tool in the proofs of self-adjointness of strongly singular potentials (see, for instance, [8], [12] and [2]). In a separate paper [13] we discuss the implications of our results to this problem.

Relative to the two-body problem, the N-body problem presents the extra difficulty of locating the threshold (the infimum of the essential spectrum of the N-body Hamiltonian with center of mass motion removed). However, if the threshold as given by Hunziker's theorem [5], is defined by a two-cluster break up we can extend the results of Simon [1] concerning sufficient conditions for the existence of infinitely many bound-states.

This paper is organized as follows. In section II we prove lemmas 1 and 2 and discuss the 2-body problem. In section III the N-body problem is briefly discussed.

II. THE TWO-BODY PROBLEM: FINITENESS AND INFINITUDE

A general proof of inequalities of type (1.7) can be found in [8]. For completeness we present the following simple

Proof of Lemma 1:-

Let $\psi(r) = g(r)\varphi(r)$, where $g^2 = h$. Then

$$\int (\psi')^2 a\,dr \geq \int \varphi^2 (g')^2 a\,dr + 2\int g g' \varphi\varphi' a\,dr =$$

$$= \int \psi^2 \left(\frac{g'}{g}\right)^2 a\,dr + \frac{1}{2}\int (\varphi^2)'(g^2)' a\,dr =$$

$$= \int \psi^2 v_a \, a\,dr \; .$$

q.e.d.

Proof of Lemma 2:-

1) Let us first consider the case $a(r) = 1$ and $h(r) = r$. Since for $\psi = r^{1/2}\varphi$

$$\int (\psi')^2 dr = \left\{ 1 + \frac{\int (\varphi')^2 r\, dr}{\int \frac{\varphi^2}{r} dr} \right\} \int \psi^2 v_a\, dr ,$$

it is enough to show the existence of an infinite sequence $\{\varphi_n \in C_0^\infty(\mathbb{R}_+ \setminus S_a)\}_{n\geq 1}$ such that

$$\frac{\int (\varphi_n')^2 r\, dr}{\int \frac{\varphi_n^2}{r} dr} < \varepsilon . \tag{2.1}$$

The ℓ.h.s. of (2.1) is scale invariant, i.e.,

$$\frac{\int (\varphi_\alpha')^2 r\, dr}{\int \frac{\varphi_\alpha^2}{r} dr} = \frac{\int (\varphi')^2 r\, dr}{\int \frac{\varphi^2}{r} dr} ,$$

where $\varphi_\alpha(r) = \varphi(\alpha r)$. It is, therefore, sufficient to find just one $\varphi \in C_0^\infty(\mathbb{R}_+ \setminus S_a)$ satisfying (2.1) and the infinite sequence $\varphi_n = \varphi(\alpha_n r)$ will be generated by suitably choosing α_n to make the supports disjoint. A possible choice of φ is, as given in [9],

$$\varphi(r) = \begin{cases} 0 , & r \leq R_0 \\ \rho(r - R_0) , & R_0 \leq r \leq R_0 + 1 \\ 1 , & R_0 + 1 \leq r \leq R_0 + N \\ \rho\left(1 - \frac{r - R_0}{N}\right), & R_0 + N \leq r \leq R_0 + 2N \\ 0 , & r \geq R_0 + 2N \end{cases}$$

with $R_0 > \max_{r \in S_a} r$ and $\rho \in C^\infty(\mathbb{R}_+)$ with $\rho(r) = 0$ if $0 \leq r \leq \frac{1}{4}$, $\rho(r) = 1$ if $r \geq \frac{3}{4}$. Since $\lim_{N\to\infty} \left(\int (\varphi')^2 r\, dr \Big/ \int \frac{\varphi^2}{r} dr \right) = 0$, it is enough to take N sufficiently large to verify (2.1).

2) Let now $\tilde{\psi}(r) = \psi(h(r))$. Then

$$\int (\tilde{\psi}')^2 a\, dr = \int (\psi')^2 dr$$

and

$$\int (\tilde{\psi})^2 v_a\, a\, dr = \frac{1}{4} \int \frac{\psi^2}{r^2} dr .$$

Taking then $\tilde{\psi}_n = \psi_n \circ f$ with ψ_n as given in part 1) makes the proof complete.

q.e.d.

Remarks. The assumption $\lim_{r\to\infty} h(r) = \infty$ is used to guarantee that the functions $\tilde{\psi}_n(r) = \psi_n(h(r))$ are not identically zero. It is not the best possible assumption for the result is still true if $a(r) = r^n$, $n \geq 1$. However, some assumption on $a(r)$ is required as the result is false if $a(r)h(r) = r^n$, $n \geq 1$.

We now describe how, starting with $a_0 = r$, it is possible to generate an infinite chain of logarithmic corrections to the "Uncertainty Principle" as described by Lemmas 1 and 2.

Let $a_n(r) = a_{n-1}(r)\,\ln_{(n)} r$, n = 1,2,... .

A straightforward computation gives, for all $\psi \in C_0^\infty(\mathbb{R}_+ \setminus S_{a_n})$,

$$U_{a_n}\left(-\frac{1}{a_n}\frac{d}{dr}a_n\frac{d}{dr}\right)U_{a_n}^{-1}\psi = U_{a_{n-1}}\left(-\frac{1}{a_{n-1}}\frac{d}{dr}a_{n-1}\frac{d}{dr} + v_{a_{n-1}}\right)U_{a_{n-1}}^{-1}\psi \,, \tag{2.2}$$

with U_a as given in the introduction. Therefore, applying Lemmas 1 and 2 to a_n obtains

<u>Lemma 3</u>. Let $v_k(r)$ be given by

$$v_0(r) = -\frac{(\nu-2)^2}{4r^2} \tag{2.3a}$$

$$v_k(r) = v_{k-1}(r) - \frac{1}{\left(2r\prod_{n=1}^{k}\ln_{(n)} r\right)^2}\,,\quad k=1,2,\dots \,. \tag{2.3b}$$

Then:

(a) For $\psi \in C_0^\infty(\mathbb{R}_+ \setminus S_{a_k})$

$$\int(\psi')^2\,dr \ge \int \psi^2 v_k\,dr \tag{2.4}$$

and

(b) For $\varepsilon > 0$, there exists an infinite sequence of non-zero functions, with disjoint supports, $\{\psi_n \in C_0^\infty(\mathbb{R}_+ \setminus S_{a_k})\}$ such that

$$\int(\psi_n')^2\,dr \le \int(\psi_n)^2\left[v_k - \frac{1+\varepsilon}{\left(2r\prod_{n=1}^{k}\ln_{(n)} r\right)^2}\right]dr \,.$$

One of the main ingredients in our discussion below is the so called "Min-Max Principle": Let H be a self-adjoint operator in Hilbert space with quadratic form domain Q(H), and for n = 1,2,... let

$$\mu_n(H) = \sup_{\varphi_1,\dots,\varphi_{n-1}}\ \inf_{\substack{\psi\in[\varphi_1,\dots,\varphi_{n-1}]^\perp \\ \|\psi\|=1,\ \psi\in Q(H)}} (\psi, H\psi)\,, \tag{2.6}$$

where $[\varphi_1,\dots,\varphi_{n-1}]^\perp$ indicates the orthogonal complement of the subspace generated by $\varphi_1,\dots,\varphi_{n-1}$. Then, for each n, either

(a) there are n eigenvalues (counting multiplicities) below the bottom of the essential spectrum, and $\mu_n(H)$ is the n-th eigenvalue counting multiplicity in increasing order

or

(b) μ_n is the bottom of the essential spectrum, and in this case, $\mu_n = \mu_{n+1} = \dots$ and there are at most (n-1) eigenvalues (counting multiplicity) below μ_n.

We are now prepared to state and prove our main results.

<u>Theorem 1</u>. Let V be a Kato potential in $L^2(\mathbb{R}^\nu)$, $\nu = 1,2,3$, such that for some $R_0 > 1$ and $\varepsilon > 0$,

$$V(x) \le v_k(r) - \frac{1+\varepsilon}{\left(2r\prod_{n=1}^{k+1}\ln_{(n)} r\right)^2}\,,\quad \begin{array}{l} k=0,1,\dots \text{ if } \nu \ne 2 \\ k=1,2,\dots \text{ if } \nu = 2 \end{array}$$

Then, the operator $[-\Delta+V]$ has infinitely many negative eigenvalues.

Proof. By the min-max principle, it is sufficient to exhibit an infinite sequence $\{\psi_n \in Q(-\Delta+V)\}_{n\geq 1}$, with disjoint supports, such that $(\psi_n, [-\Delta+V]\psi_n) < 0$. The existence of such a sequence follows from Lemma 3.

q.e.d

Theorem 2. Let V be a Kato potential in $L^2(\mathbb{R}^\nu)$, $\nu = 1,2,3$, such that, for $R_0 > 1$, $c < 1$ and k,

$$V(x) \geq c\, v_k(r) \qquad \text{if} \qquad r > R_0$$

where $k = 0,1,\ldots$ if $\nu \neq 2$ and $k = 1,2,\ldots$ if $\nu = 2$.
Then $[-\Delta+V]$ has at most finitely many negative eigenvalues.

Proof. We first decompose our operator into

$$-\Delta + V = (-c\Delta + V\chi_2) + (-(1-c)\Delta + V\chi_1),$$

where $\chi_1 \in C_0^\infty$, $\chi_1(\vec{x}) = 1$ if $r \leq R_0$, $0 \leq \chi_1 \leq 1$
and $\chi_2(x) = 1 - \chi_1(x)$.

From a simple application of the min-max principle, it follows that if both operators $A = -(1-c)\Delta + V\chi_1$ and $B = -c\Delta + V\chi_2$ (which are essentially self-adjoint in the same domain and have the same essential spectrum) have finitely many negative eigenvalues then the same holds for $-\Delta+V = A+B$ (see for instance [2], vol. IV exercise 129 pg 379).

That the operator A has finitely many negative eigenvalues is a standard result since the potential $V\chi_1$ has compact support (see for instance [2], vol. IV exercise 20 pg 366). On the other hand, by assumption, $B > c\,(-\Delta + \chi_2 v_k)$ and

it is therefore sufficient to show that the operator $-\Delta + \chi_2 v_k$ has finitely many negative eigenvalues. If $\nu \geq 2$ it is sufficient to consider the operator $-\Delta + \chi_2 v_k$ restricted to the subspace $\mathcal{H}_0$ of spherically symmetric functions since in $\mathcal{H}_0^\perp$ the operator is positive! The restriction to $\mathcal{H}_0$ is given by the operator

$$H_k = \left\{ -\frac{1}{r^{\nu-1}} \frac{d}{dr} r^{\nu-1} \frac{d}{dr} + \chi_2 v_k(r) \right\}.$$

For $\nu=1$ we consider the operator $\left(-\frac{d^2}{dx^2}\right)_D + \chi_2 v_k$, with Dirichlet boundary conditions on $\pm R_0$.

For $\nu = 2,3$ a similar argument applies for the operator H_k restricted to $\mathcal{H}_0$, thus concluding the proof.

q.e.d.

Remarks. From the proofs it is clear that the finiteness or infinitude is controlled by the following limits:

$$\mu_0 = \lim_{r\to\infty} (2r)^2 V(x), \ \ldots, \ \mu_k = \lim_{r\to\infty} \left\{ \left(2r \prod_{n=1}^{k} \ln_{(n)} r\right)^2 [V(x) - v_{k-1}(r)] \right\}.$$

Indeed finiteness is implied by

$$\mu_0 = \cdots = \mu_{k-1} = -1,\ \mu_k > -1 \quad \text{for some } k \geq 0 \quad \text{if} \quad \nu \neq 2$$

$$\mu_1 = \cdots = \mu_{k-1} = -1,\ \mu_k > -1 \quad \text{for some } k \geq 1 \quad \text{if} \quad \nu = 2,$$

whereas infinitude is guaranteed by

$u_0 = \cdots = u_{k-1} = -1, \mu_k < -1$ for some $k \geq 0$ if $\nu \neq 2$

$u_1 = \ldots = u_{k-1} = -1, \mu_k < -1$ for some $k \geq 1$ if $\nu = 2$.

III. THE N-BODY PROBLEM: INFINITUDE

This section constitutes a sort of appendix of section 3 of Barry Simon's work [1]. Therefore we shall not give all the details and instead we shall be rather sketchy.

Let us consider a system of N-particles, with masses m_i, $i = 1,\ldots,N$, in $\nu = 1, 2$ or 3 dimensions, interacting via two-body Kato potentials $V_{ij}(\vec{r}_i - \vec{r}_j)$. The Hamiltonian H_N, after removal of the center of mass motion,

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i<j}^{N} V_{ij}(\vec{r}_i - \vec{r}_j) - \frac{(\Sigma p_i^2)^2}{2(\Sigma m_i)} ,$$

has the infinum Σ of its essential spectrum given by Hunziker's theorem [5]:

$$\Sigma = \min_{\substack{D_1 \cap D_2 = \phi \\ D_1 \cup D_2 = \{1,\ldots,N\}}} [\sigma_{D_1} + \sigma_{D_2}] ,$$

where σ_D = infinum spectrum H_D; here H_D denotes the Hamiltonian of the cluster $D \subset \{1,\ldots,N\}$, with center of mass kinetic energy removed. If $\sigma = \sigma_{D_1} + \sigma_{D_2}$ and H_{D_1} and H_{D_2} have discrete ground states at the bottom of their spectra we say, after [1], that the system has a "two-cluster continuum limit".

It should be remarked that there is a number of situations for which it can be proved that the system has a "two-cluster continuum limit", namely: a) for $\nu = 1,2$ a sufficient condition is that $\int V_{ij}(\vec{x}) d^\nu x < 0$ [6]; b) for $\nu=3$ a sufficient condition is that V_{ij}'s are purely attractive and hold a bound state [7].

As in [1], if we are in the two-cluster limit case, sufficient conditions for infinitude can be obtained by reducing the analysis to that of an effective two-body problem.

Theorem 3. Let V_{ij} be Kato potentials that are C^∞ functions on an open set of $\mathbb{R}^\nu$ whose complement has zero measure and let Σ be given by a two cluster break up (D_1,D_2), with reduced mass $\mu_{D_1 D_2} = \left(\frac{1}{\sum_{i\in D_1} m_i} + \frac{1}{\sum_{j\in D_2} m_j}\right)^{-1}$. Denoting by $\vec{R}$ the relative coordinate of the center of masses of clusters D_1 and D_2, if the potential

$$\tilde{V}_{D_1 D_2}(\vec{R}) = 2\mu_{D_1 D_2} \sum_{\substack{i \in D_1 \\ j \in D_2}} V_{ij}(\vec{R})$$

satisfies the assumptions of theorem 1, then H_N has infinitely many eigenvalues below Σ.

Remark. We believe that this theorem holds for Kato potentials without that extra smoothness assumption.

Proof. Since $H_N = H_{D_1} + H_{D_2} + V_{D_1 D_2} - \frac{1}{2\mu_{D_1 D_2}} \Delta_{\vec{R}}$, where $V_{D_1 D_2} = \sum_{\substack{i\in D_1 \\ j \in D_2}} V_{ij}(\vec{x}_i - \vec{x}_j)$ is the intercluster potential, for

$\psi = \psi_{D_1} \psi_{D_2} \phi$ we have

$$(\psi, H_N \psi) = E_{D_1} + E_{D_2} + \left(\phi, \left[-\frac{1}{2\mu_{D_1D_2}} \Delta_{\vec{R}} + \bar{V}\right] \phi\right) \quad ,$$

where $\bar{V}(\vec{R}) = \sum_{\substack{i \in D_1 \\ j \in D_2}} (\psi_{D_1} \psi_{D_2}, V_{ij}(\vec{x}_i - \vec{x}_j) \, \psi_{D_1} \psi_{D_2})$

is the effective intercluster potential when the clusters D_1 and D_2 are in their bound states ψ_{D_1} and ψ_{D_2}, respectively, with corresponding energies E_{D_1} and E_{D_2}.

The proof of the theorem is completed by the following generalization of Proposition 5 in [1]:

Lemma 4. Let ψ_{D_i} be a bound state of H_{D_i}, a k_i-body system with Kato potentials that are C^∞ functions on an open set of $\mathbb{R}^\nu$ whose complement has zero measure. Let V_{ij} be Kato potentials such that for some $\gamma \leq 2$ and $\ell \geq 1$

$$\overline{\lim_{r\to\infty}} \left(2r \prod_{n=1}^{\ell} \ln_{(n)} r\right)^{\gamma} \left[V_{ij}(\vec{x}) - v_{\ell-1}(r)\right] \leq C_\ell \quad ,$$

v_ℓ given by (2.3). Let

$$\bar{V}_{ij}(\vec{R}) = \int |\psi_{D_1}(\vec{r}_1)|^2 |\psi_{D_2}(\vec{r}_2)|^2 \, V_{ij}(\vec{r}_{ij}(\vec{R}, \vec{r}_1, \vec{r}_2)) \, d^{\nu(k_1-1)} r_1 \, d^{\nu(k_2-1)} r_2 \quad ,$$

where $\vec{r}_{ij}(\vec{R}, \vec{r}_1, \vec{r}_2)$ is the distance between particles $i \in D_1$ and $j \in D_2$, in terms of the internal coordinates $\vec{r}_1$ ($\vec{r}_2$) of D_1 (D_2) and the distance $\vec{R}$ between the centers of mass of D_1 and D_2. Then

$$\overline{\lim_{R\to\infty}} \left(2R \prod_{n=1}^{\ell} \ln_{(n)} r\right)^{\gamma} \left[\bar{V}_{ij}(\vec{R}) - v_{\ell-1}\right] \leq C_\ell \quad .$$

Proof. The proof follows by repetition of the steps in [1, Proposition 5] having in mind that the extra smoothness assumption on the potentials ensures that the function

$$\rho(\vec{r}_0) = \int d^{\nu(k_1+k_2-\nu)} r \, |\psi_{D_1}(\vec{r}_1)|^2 \, |\psi_{D_2}(\vec{r}_2)|^2$$

(with integration over all coordinates but $\vec{r}_0 : \vec{r}_{ij}(\vec{R}, \vec{r}_1, \vec{r}_2) = \vec{R} - \vec{r}_0$) decays faster than any power:

$$\sup_{\vec{r}_0} |\rho(\vec{r}_0)(1 + r_0^n)| < \infty \quad ,$$

for all n. This is a result by Hunziker [5].

q.e.d.

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