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ABSTRACT

We construct a hierarchical model for 2-d Coulomb gases displaying a line stable of fixed points describing the Kosterlitz-Thouless phase transition. For Coulomb gases corresponding to $\mathbb{Z}_N$- models these fixed points are stable for an intermediate temperature interval.

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The Kosterlitz-Thouless phase transition [1,2] occurs in a class of 2-dimensional systems like the plane rotator, Coulomb gases and $\mathbb{Z}_N$ models, $N \gg 1$ [3,4,12]. For the plane rotator it is characterized by a change of exponential to power law decay of correlation function as the temperature is lowered. The physics of this transition is explained by the competition of the self-energy and entropy of the defects (vortices) occuring in the system [1,2,3].

Renormalization group (R.G.) methods have been employed to discuss the phenomenon. Usually, calculations with R.G. make an approximation of disregarding non-local contributions to the transformed Hamiltonian. In the so called hierarchical models no non-local terms appear and therefore the above approximation scheme is exact.

For the Kosterlitz-Thouless phase transition however, the only existing hierarchical model is the one for which the so-called Migdal-Kadanoff R.G. formula is exact. The trouble with this approach is that the Migdal-Kadanoff recursion formulae, as seen numerically by José et al. [3] and rigorously proved by Ito [7], have no stable fixed point other than the $T = \infty$ one.

In this letter we describe a 2-d hierarchical model such that the line of fixed points corresponding to massless gaussian theories is, for $0 < T \leq T_c < \infty$, (globally) stable against a class of perturbations that include Coulomb gas-type of interactions. Therefore those Coulomb gases have, for $T < T_c$, an asymptotic behavior of massless gaussian field. This is the Kosterlitz-Thouless phase transition.

Our model incorporates the ideas of Wilson [8], as formulated by Gawedzki and Kupiainen [9,10] of decomposing the field operator ϕ into a sum of two fields ψ and ξ describing the block-spin and fluctuation variables respectively. It is described as follows.

The starting point is the hierarchical 2-dimensional massless gaussian field $\phi(x)$ in a cubic lattice Z^2, defined by the two-point function:

$$\langle e^{i\phi(x)} e^{-i\phi(y)}\rangle = \int d\mu_{\beta G_H}\, e^{i(\phi(x)-\phi(y))} = L^{-\frac{\beta}{2\pi}[N_L(x,y)-N_L(0,0)]} \tag{1}$$

The measure $d\mu_{\beta G_H}(\phi)$ is formally given by $\exp\left\{-\frac{1}{2\beta}\sum_{x,y}\phi(x)G_H^{-1}(x,y)\phi(y)\right\}\prod_x d\phi(x)$ where $G_H(x,y) = -\frac{1}{2\pi}N_L(x,y)\ln L$ plays the role of a Green's function for the "hierarchical laplacean".

Here $L>1$ is an integer representing a scale parameter in the model and $N_L(x,y)$, the "hierarchical distance" between x and y, is the smallest positive integer N such that $[L^{-N}x] = [L^{-N}y]$, ($[Z]$ denotes the vector formed with the integer part of the components of $Z \in \mathbb{R}^2$), and so $N_L(0,0)=1$. Our choice of the free hierarchical covariance is made as to guarantee that the asymptotic behavior of correlation function of exponentials of the field $\phi(x)$, are given by:

$$\langle e^{i(\phi(x)-\phi(y))}\rangle \underset{|x-y|\to\infty}{\sim} c\,|x-y|^{-\frac{\beta}{2\pi}} \tag{2}$$

Notice that this differs from the usual formulation [10,11] of the hierarchical models (for $d>2$) where the correlation function of the fields themselves are asymptotically equal to that of a usual free massless theory.

Following Gawedzki and Kupiainen [11] we introduce the orthogonal decomposition:

$$\phi(x) = \psi\left(\left[\tfrac{x}{L}\right]\right) + \xi\left(L\left[\tfrac{x}{L}\right]\right) \tag{3}$$

where ψ is a gaussian field with two-point function:

$$\langle e^{i(\psi(x)-\psi(y))}\rangle = \langle e^{i(\phi(x)-\phi(y))}\rangle \tag{4}$$

and ξ is the gaussian "fluctuation field" determined by

$$\langle e^{i(\xi(x)-\xi(y))}\rangle = \int e^{i(\xi(x)-\xi(y))}\,d\nu(\xi) = \begin{cases} 1 & \text{if } x=y \\ L^{-\frac{\beta}{2\pi}} & \text{if } x\neq y \end{cases} \tag{5}$$

$$\langle e^{it\xi(x)}\rangle = L^{-\frac{\beta}{4\pi}t^2}\ .$$

Notice that $\xi(x)$ is independent of $\xi(y)$ if $x\neq y$ and that the contribution of the $\xi(L[\frac{x}{L}])$ is constant when x varies in a given block of side L.

The class of models we are going to consider is obtained by a local perturbation Λ of the gaussian-measure $d\mu_{\beta G_H}$:

$$\langle F(\phi)\rangle_\Lambda = Z^{-1}\int F(\phi)\,\Lambda(\phi)\,d\mu_{\beta G_H}(\phi) = Z^{-1}\int F(\phi)\prod_{x\in \mathbb{Z}^2}\lambda(\phi(x))\,d\mu_{\beta G_H}(\phi) \tag{6}$$

The renormalization group transformation $\Lambda \to R_L\Lambda$ is defined by integration over the "fluctuation variables" ξ:

$$(R_L\Lambda)(\psi) = \int d\nu(\xi)\ \Lambda(\phi) \tag{7}$$

and it corresponds to the usual block spin transformation. This transformation is in fact of a local nature. It follows from (3) and (5) that

$$(R_L\Lambda)(\psi) = \prod_{x\in\mathbb{Z}^2} (r_L\lambda)(\psi(x)) \tag{8}$$

where

$$(r_L\lambda)(\psi) = \int d\nu(\xi)\ [\lambda(\psi+\xi)]^{L^2} \tag{9}$$

By construction the free theory, $\lambda=1$, is a fixed point of the transformation r_L. We are interested in analysing the stability of this fixed point with respect to a special class τ of local perturbation λ. The choice of this class has to meet two requirements: 1) it must contain $\lambda(\phi) = \exp\{Z\cos\phi\}$ as this represents the "standard" Coulomb gas and 2) it must be closed under the renormalization group transformation (9), i.e., $r_L\lambda \in \tau$ if $\lambda \in \tau$.

A minimal choice of τ fulfilling 1) and 2) is:

$$\lambda(\phi) = \sum_{q\in\mathbb{Z}} z_q\ e^{iq\phi} \tag{10}$$

where $Z \in \ell^1$ i.e. $\sum_{q\in\mathbb{Z}} |z_q| < \infty$, and $z_q = z_{-q}$.

If we write:

$$(r_L\lambda)(\phi) = \sum_{q\in\mathbb{Z}} z'_q\ e^{iq\phi} \tag{11}$$

An explicit computation shows:

$$z'_q = L^{-\frac{\beta}{4\pi}q^2} \sum_{\substack{q_1,\dots,q_{L^2}\in\mathbb{Z} \\ \sum_{i=1}^{L^2} q_i = q}} z_{q_1}\cdots z_{q_{L^2}} \tag{12}$$

in particular $Z' \in \ell^1$.

The stability of the fixed point $\lambda_0(\phi) = 1$ corresponding $z_q^{(0)} = \delta_{q_0}$ can be analyzed by linearizing the R.G. transformation (12) around $z_q^{(0)}$. The linearized transformation $\underline{Z}' = A\underline{Z}$ is given by the diagonal matrix:

$$A_{qq'} = \begin{cases} \delta_{qq'}\ L^{2-\frac{\beta}{4\pi}q^2} & \text{if } q \neq 0 \\ 0 & \text{if } q = 0 \end{cases} \tag{13}$$

Therefore if $\beta > \beta_c = 8\pi$ the eigenvalues γ_n of A satisfy $|\gamma_n| < 1$ and the fixed points is stable.

In fact it is not difficult to show that the fixed point is globally attractive, i.e., for any $\lambda \in \tau$, $\lim_{N\to\infty}(r_L^N\lambda) = \lambda_0$ if $\beta > \beta_c$. Full mathematical detail will be presented elsewhere [14].

For the $\mathbb{Z}_N$-models, $N \gg 1$, the Kosterlitz-Thouless phenomenon is characterized [3,12] by the existence of an intermediate temperature interval $I_N = [\underline{\beta_N}, \overline{\beta_N}]$ such that for $\underline{\beta_N} < \beta < \overline{\beta_N}$ the correlation function decay polynomially. These models can be show [12] to be equivalent to two interacting Coulomb gases with integer charges m and n at temperatures β and $\beta' = \frac{4\pi^2N^2}{\beta}$ respectively. A simplified hierarchical version of this system is given by probability distribution:

$$d\mu_{\beta G_H}(\phi) \prod_{x\in\mathbb{Z}^2} \lambda^{(N)}(\phi(x)) \tag{14}$$

where

$$\lambda^{(N)}(\phi) = \sum_{m,n\in\mathbb{Z}} g_{mn}\ e^{i(m+\frac{2\pi N}{\beta}n)\phi} \tag{15}$$

Now if $g_{mn} = g_{nm}$, $d\mu_{\beta G_H} \lambda(\phi)$ is invariant under $\beta \to \frac{(2\pi)^2 N^2}{\beta}$, and this expresses the sel-duality typical of Z_N-symmetric models [12].

A renormalization group transformations (9) acts on a model given by $g = \{g_{mn}, m, n \in \mathbb{Z}\}$ through (15) transforming it on another model of the same class given by a different set $g' = \{g'_{mn}, m, n \in \mathbb{Z}\} = r_L g$:

$$g'_{mn} = L^{-\frac{\beta}{4\pi}\left(m+\frac{2\pi N}{\beta}n\right)^2} \sum_{\substack{m_1,\ldots,m_{L^2} \\ \sum_{i=1}^{L^2} m_i = m}} \sum_{\substack{n_1,\ldots,n_{L^2} \\ \sum_{i=1}^{L^2} n_i = n}} \left[\prod_{i=1}^{L^2} g_{m_i n_i}\right] \tag{16}$$

The linearized transformation is given by the matrix:

$$B_{m'n';mn} = \begin{cases} L^{2-\frac{\beta}{4\pi}\left(m+\frac{2\pi N}{\beta}n\right)^2} \delta_{mm'} \delta_{nn'} & , m \neq 0 \text{ and(or) } n \neq 0 \\ 0 & , m = 0 \text{ and } n = 0 \end{cases} \tag{17}$$

Therefore we have stability if and only if $\beta > 8\pi \equiv \underline{\beta}_N$ and $\beta < \frac{\pi N^2}{2} \equiv \overline{\beta}_N$. The two conditions are incompatible if $N < 4$ and for $N > 4$ $\overline{\beta}_N > \underline{\beta}_N$ and there will be a soft intermediate phase for $\beta \in (\underline{\beta}_N, \overline{\beta}_N)$.

Finally we should remark that our results and techniques admit a natural extension in order to cover U(1) and Z_N-lattice gauge theories in 4-dimensions [14] as the deconfining phase transition of these models are of the same nature as the Kosterlitz-Thouless phenomenon in 2-dimensions [12,13,15,16].

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