UNIVERSIDADE DE SÃO PAULO

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IFUSP/P-558

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Novembro/1985

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Abstract

Using the equivalence with a derivative coupling model, mass perturbation in the Thirring model is investigated. We show that, for $4\pi(2-\sqrt{3}) < \beta^2 < 8\pi$ all ultraviolet divergences cancel. Finite composite operators are constructed in this range. Ward identities and equations of motion are discussed.

I. Introduction

The usual approach to perturbative studies of models consists in separating the Lagrangian, L, describing the system in two pieces, L_o and L_{Int}

$$L = L_o + L_{Int} \tag{I.1}$$

with L_o a free field Lagrangian and L_{Int} containing all relevant interactions. This division is dictated solely by our ignorance and inability to produce solutions for more general field equations. It is certainly desirable to have at one's disposal a perturbative scheme with L_o already incorporating as many symmetries as possible. Of course, if a project has a too broad scope it is probably untenable. We have therefore limited our attention to mass perturbation around scale invariant theories. Barring the uninteresting and trivial case of perturbation of free field theories this brings us immediately to the context of some soluble two dimensional models. A prominent member of this class is the Thirring model which has so much contributed to the development of ideas in Field Theory [1]. In particular the reader should recall the amazing equivalence of this fermionic model with a bosonic theory, the sine-Gordon model [2]. To fix a notation, let k be the Thirring model coupling constant in Klaiber's definition [1]. Then the sine-Gordon parameter, β, which appears in the interaction $\cos(\beta\varphi)$, is related to k by

$$\frac{k}{\pi} = \left(\frac{4\pi}{\beta^2}\right)^{1/2}\left(1 - \frac{\beta^2}{4\pi}\right) \tag{I.2}$$

Attractive and repulsive regions correspond to $\beta^2 < 4\pi$ ($k < 0$) and $\beta^2 > 4\pi$ ($k > 0$). $\beta^2 = 4\pi$ corresponds to a free fermion theory.

The Thirring model has other interesting connections. It is also equivalent, in a sense to be made precise later, to a derivative coupling model describing two massless scalar fields, φ_1 and φ_2, interacting with a massless spinor field, ψ, via the interaction Lagrangian

$$L_{Int} = g_1 (\bar{\psi}\gamma^\mu\psi)\partial_\mu\varphi_1 + g_2(\bar{\psi}\gamma^\mu\gamma^5\psi)\partial_\mu\varphi_2 \qquad (I.3)$$

The model (I.3) will be called Derivative Coupling Model, DC model, for conciseness. If $g_2 = 0$ it becomes a model studied by Schroer.[3] The massive model with $g_1 = 0$ was considered by Rothe and Stamatescu.[4]

To be equivalent to the Thirring model, the couplings g_1 and g_2 can not be independent, but are related to Klaiber's constant k by

$$g_2^2 = k\left[\left(1+\left(\frac{k}{2\pi}\right)^2\right)^{1/2} + \frac{k}{2\pi} \right] \qquad (I.4)$$

$$\left(1+\frac{g_1^2}{\pi}\right)\left(1+\frac{g_2^2}{\pi}\right) = 1 \qquad (I.5)$$

Mass perturbation around a massless theory is plagued by severe infrared divergences. In such situation, one should attempt to make partial resummations to achieve finiteness. But, without a guiding principle, this is a hopeless task. We shall therefore adopt an infrared regulator before proceeding. A detailed discussion of the ultraviolet behaviour is then done and the following result obtains

1. For $\beta^2 < 4\pi$ the more divergent contributions are precisely those of the unperturbed model. We found that only for $4\pi(2 - \sqrt{3}) < \beta^2$ the Thirring Green functions are welll defined.

Parenthetically, this does not mean that the Thirring model is pathological for β^2 bellow $4\pi(2 - \sqrt{3})$; the Wightman functions as given by Klaiber are, for example, well defined for all values of k. The value $\beta^2 = 4\pi(2 - \sqrt{3})$ is the point where the two point Green function becomes singular as a distribution. We could still continue analitically beyond this value, decreasing β^2, but this process will lead to more and more divergent Green functions. Finally, at $\beta^2 = 4\pi(4 - (15)^{1/2})$ all Green functions will become divergent and no continuation to lower values will be possible. We also mention that in the interval $4\pi(2 - \sqrt{3}) < \beta^2 < 4\pi$ the only singularities are volume divergences which, as usual, cancel between numerator and denominator in the Gell Mann Low formula.

2. For $8\pi > \beta^2 > 4\pi$ there are some additional divergences associated with vacuum bubble diagrams. These are again canceled by the denominator of the Gell Mann Low formula.

3. We also dicuss the construction of composite operators. In particular, we verify a conjecture by Swieca for the definition of the mass operator $N[\bar{\psi}\psi]$[5]. It is found that a well defined operator is obtained just by doing a subtraction of the vacuum expectation value besides the usual Wick ordering prescription.

For $\beta^2 > 8\pi$ the theory is unrenormalizable and some drastic change in the approach would be necessary.

The paper is organized as follows

In section II the DC model is introduced, firstly at the classical level. We then show that the fermionic Green functions of the model are, for certain identification of the coupling constants, equal to those of the massless Thirring model. The section ends with a brief discussion of composite objects as the fermionic current and the mass operator. Section III begins the discussion of mass perturbation by giving the rules to construct the relevant amplitudes. An infrared cutoff is introduced and the degree of superficial divergence of an arbitrary amplitude is established. The UV behaviour is extensively analysed in section IV where we also discuss the modifications, if any, in the case of composite operators. Equations of motion and Ward identities are discussed in section V. Some remarks concerning the elimination of the infrared cutoff are presented in the conclusions.

II.A Derivative Coupling Model

From a techical point of view the study of mass perturbation in the Thirring model can be greatly simplified if one takes advantage of the equivalence of this theory with the derivative coupling model specified by

$$L = \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi + \frac{1}{2}(\partial_\mu\varphi_1)^2 + \frac{1}{2}(\partial_\mu\varphi_2)^2 + g_1(\bar{\psi}\gamma^\mu\psi)\partial_\mu\varphi_1 + g_2(\bar{\psi}\gamma^\mu\gamma^5\psi)\partial_\mu\varphi_2 \qquad \text{(II.1)}$$

At the classical level the equations of motion derived from such Lagrangian are

$$\partial^2\varphi_1 = -g_1\partial_\mu(\bar{\psi}\gamma^\mu\psi) \qquad \text{(II.2)}$$

$$\partial^2\varphi_2 = -g_2\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) \qquad \text{(II.3)}$$

$$i\partial\!\!\!/\psi = -g_1(\partial_\mu\varphi_1)\gamma^\mu\psi - g_2(\partial_\mu\varphi_2)\gamma^\mu\gamma^5\psi \qquad \text{(II.4)}$$

Now, as $\gamma^\mu\gamma^5 = \varepsilon^{\mu\nu}\gamma_\nu$, $\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we could use (II.2) and (II.3) to reconstruct the current

$$\bar{\psi}\gamma^\mu\psi = -\frac{1}{g_1}\partial^\mu\varphi_1 - \frac{1}{g_2}\tilde{\partial}^\mu\varphi_2 \quad , \quad \tilde{\partial}^\mu = \varepsilon^{\mu\nu}\partial_\nu \qquad \text{(II.5)}$$

Comparing this expression with the equation of motion of the Thirring model

$$i\partial\!\!\!/\psi = -g(\bar{\psi}\gamma^\mu\psi)\gamma_\mu\psi \qquad \text{(II.6)}$$

we see that with the choice $g_1^2 = -g_2^2 = -g$, the two models have identical fermionic sectors.

The content of the model (II.1) is actually trivial. As both vector and axial vector currents are conserved, φ_1 and φ_2 turn out to be free fields. Moreover, from (II.4) one easily get $\psi = \mathrm{Exp}(ig_1\varphi_1 + i\gamma^5 g_2\varphi_2)\psi_0$ with ψ_0 a free massless Dirac field.

The next step is the quantization. It is clear that the equivalence will continue to hold if the same quantization prescription is adopted for both models. At this point it may be instructive to stress a very fundamental difference between the classical and the quantum descriptions of a field theory. The classical equation of motion does not specify a model because quantum fluctuations make the interaction terms undefined. To promote these formal expressions to the status of bona fide quantum operators requires detailed information about the short distance behaviour of a product of fields. In general terms, this implies that field equations and their solutions must be given simultaneously to, self consistently, caracterize the theory. In our case we suppose that φ_1 and φ_2 will still be free fields. However, since they are massless an infrared regulator is necessary to achieve finiteness. The infrared regulated two point functions are

$$\langle T\varphi_1(x)\,\varphi_1(0)\rangle = \langle T\varphi_2(x)\,\varphi_2(0)\rangle = D_F(x) = -(1/4\pi)\,\lg\mu^2(-x^2+i\epsilon) \tag{II.7}$$

where $D_F(x)$ satisfies $\partial^2 D_F(x) = -i\,\delta(x)$. Because of the infrared cutoff the Hilbert space of the states constructed from the fields φ_1 and φ_1 does not have a positive definite norm. In spite of this, exponentiated fields $:\mathrm{Exp}\; i\alpha\varphi(x):$ are in a good shape, provided a certain charge conservation law is obeyed. The precise statement concerning the last remark is that positivity holds in the subspace reconstructed from Wightman's functions satisfying a charge conservation law:(6)

$$\langle T{:}\mathrm{Exp}\, i\alpha_1\varphi(x_1){:}\,{:}\mathrm{Exp}\, i\alpha_2\varphi(x_2){:}\ldots{:}\mathrm{Exp}\, i\alpha_n\varphi(x_n){:}\rangle = \tag{II.8}$$

$$= \mathrm{Exp}\Big[-\sum_{i<j}\alpha_i\alpha_j\, D_F(x_i - x_j)\Big]\,\delta_{\sum\alpha_i,\,0}$$

Thus, at least for small g_1 and g_2, the fermionic sector could be described by the field

$$\psi_T = {:}\mathrm{Exp}\,(ig_1\varphi_1 + ig_2\varphi_2){:}\,\psi_0 \tag{II.9}$$

Indeed, using (II.8), the N point Green function can be computed and then compared with Klaiber's. We have

$$\langle T\psi_T(x_1)\ldots\psi_T(x_N)\bar\psi_T(y_1)\ldots\bar\psi_T(y_N)\rangle = \mathrm{Exp}\Big[\sum_{i<j} -(g_1^2 + g_2^2\,\gamma^5_{x_i}\gamma^5_{x_j})\,D_F(x_i - x_j)\Big]$$

$$\cdot\,\mathrm{Exp}\Big[\sum_{i<j} -(g_1^2 + g_2^2\,\gamma^5_{y_i}\gamma^5_{y_j})\,D_F(y_i - y_j)\Big]\cdot\mathrm{Exp}\Big[\sum_{i,j} -(-g_1^2 + g_2^2\,\gamma^5_{x_i}\gamma^5_{y_j})\,D_F(x_i - y_j)\Big]$$

$$\cdot\,\langle T\psi_0(x_1)\ldots\psi_0(x_N)\bar\psi_0(y_1)\ldots\bar\psi_0(y_N)\rangle \tag{II.10}$$

in which we should identify

$$a = g_1^2 \quad , \quad b = g_2^2$$

$$a = k\Big[-\Big(1+\big(\tfrac{k}{2\pi}\big)^2\Big)^{1/2} + \frac{k}{2\pi}\Big] \,,\quad b = k\Big[\Big(1+\big(\tfrac{k}{2\pi}\big)^2\Big)^{1/2} + \frac{k}{2\pi}\Big] \tag{II.11}$$

where k is the Thirring model coupling constant as defined by Klaiber. Note that $g_1^2 g_2^2 = -k^2$, implying that one of the g's is imaginary.

We are now in a position to write down all the operators appearing in the equation of motion in terms of φ_1, φ_2, and ψ_0. The current, for example, can be identified with

$$g_\mu = \frac{1}{k}\,(g_1\,\partial_\lambda\varphi_1 - g_2\,\tilde\partial_\lambda\varphi_2) \tag{II.12}$$

so that the field equations become

$$i\partial\!\!\!/\,\psi(x) = -\frac{k}{2}\lim_{\epsilon\to 0}\big(g^\mu(x+\epsilon)\gamma_\mu\psi(x) + \gamma_\mu\psi(x)\,g^\mu(x-\epsilon)\big) \tag{II.13a}$$

$$\partial^2\varphi_1 = \frac{k}{g_1}\,\partial_\lambda g^\mu \tag{II.13b}$$

$$\partial^2\varphi_2 = \frac{k}{g_2}\,\tilde\partial_\lambda g^\mu \tag{II.13c}$$

Composite operators can also be constructed as local limits of products of the basic fields. In particular, Johnson's(1) limiting procedure furnishes the current

$$j^\mu(x) = \frac{1}{4}\Big(1+\big(\tfrac{k}{2\pi}\big)^2\Big)^{-1/2}\lim_{\epsilon\to 0}\sum_{\epsilon,\tilde\epsilon} Z(\epsilon)\big(\bar\psi(x+\epsilon)\gamma^\mu\psi(x) - \gamma^\mu\psi(x)\,\bar\psi(x-\epsilon)\big) \tag{II.14}$$

$$Z(\epsilon) = \mathrm{Exp}\big[-(g_1^2 + g_2^2)\,D_F(\epsilon)\big]$$

which, as discussed elsewhere, differs from (II.12) by a factor containing a spurion field, i.e., a field which has no effect on the fermionic sector. For later reference, we also write the mass operator as a limiting process

$$N[\bar\psi_T\psi_T](x) = \lim_{\epsilon\to 0}\mathrm{Exp}\big[-(g_1^2 - g_2^2)D_F(\epsilon)\big]\bar\psi_T(x+\epsilon)\psi_T(x) = {:}\mathrm{Exp}\,2ig_2\gamma^5\varphi_2{:}\,{:}\bar\psi_0\psi_0{:}(x) \tag{II.15}$$

with the understanding that the γ^5 matrix acts immediately on the left of the ψ_0 field.

III. Mass Perturbation

In Klaiber's operator approach the field solution of the Thirring model is written as

$$\psi_T = {:}\mathrm{Exp}\,(i\alpha j + i\beta\gamma^5\tilde j)\,\psi_0{:} \tag{III.1}$$

where j and $\tilde j$ are the potentials of the free vector and free axial vector currents, respectively. As j, $\tilde{j}$ and ψ_0 are not independent, the study of mass perturbation may become rather cumbersome. In this respect the representation (II.9), employing independent fields is clearly superior and will be adopted from now on.

The formal study of the perturbative series can be done by defining Green functions via the Gell Mann Low formula

$$\langle T\psi(x_1)\ldots\psi(x_N)\bar{\psi}(y_1)\ldots\bar{\psi}(y_N)\rangle = \frac{\langle T\psi_T(x_1)\ldots\psi_T(x_N)\bar{\psi}_T(y_1)\ldots\bar{\psi}_T(y_N)\,\mathrm{Exp}[i\int L_{Int}\,d^2x]\rangle}{\langle T\,\mathrm{Exp}[i\int L_{Int}\,d^2x]\rangle} \tag{III.2}$$

where $L_{Int} = N[\bar{\psi}_T\psi_T]$ is the mass operator and ψ_T denotes the solution (III.1) of the Thirring model. The Feynman amplitudes are obtained by expanding the exponential and applying Wick's theorem, always keeping in mind the selection rules (II.8). For that it is useful the identity

$$:\mathrm{Exp}\,2ig_2\gamma^5\varphi:\bar{\psi}\psi = :\mathrm{Exp}-2ig_2\varphi:\bar{\psi}_1\psi_1 + :\mathrm{Exp}\,2ig_2\varphi:\bar{\psi}_2\psi_2 \tag{III.3}$$

We shall now study the ultraviolet behaviour of the integrals so constructed. To simplify the discussion this will be done explicitly in the Euclidian region. A generic amplitude I_G consists of a product of propagators of various types which, in a graphical representation, are associated with the lines of a graph G. The vertices of G are associated either with the interaction Lagrangian or with external fields. In momentum space the possible propagators are

1. Fermion propagator $\dfrac{\not{p}}{p^2}$

2. Exponentiated fields. These are of various types, depending on the contracted fields ($\sigma = \dfrac{g_2^2}{4\pi} = \dfrac{1}{4}\left(\dfrac{g^2}{4\pi} - 1\right)$; $\rho = \dfrac{g_1^2}{4\pi} = -\sigma/(1+4\sigma)$):

Propagator		Contraction	
a. $(p^2)^{-4\sigma-1}$	. . . .	$:\mathrm{Exp}\,2ig_2\varphi_2:$ with $:\mathrm{Exp}\,2ig_2\varphi_2:$	or
		$:\mathrm{Exp}-2ig_2\varphi_2:$ with $:\mathrm{Exp}-2ig_2\varphi_2:$	
b. $(p^2)^{4\sigma-1}$	. . . .	$:\mathrm{Exp}-2ig_2\varphi_2:$ with $:\mathrm{Exp}\,2ig_2\varphi_2:$	
c. $(p^2)^{-2\sigma-1}$	. . . .	$:\mathrm{Exp}\,2ig_2\varphi_2:$ with $:\mathrm{Exp}\,ig_2\varphi_2:$	or
		$:\mathrm{Exp}-2ig_2\varphi_2:$ with $:\mathrm{Exp}-ig_2\varphi_2:$	
d. $(p^2)^{2\sigma-1}$	. . . .	$:\mathrm{Exp}\,2ig_2\varphi_2:$ with $:\mathrm{Exp}-ig_2\varphi_2:$	or
		$:\mathrm{Exp}-2ig_2\varphi_2:$ with $:\mathrm{Exp}\,ig_2\varphi_2:$	
e. $(p^2)^{-\sigma-1}$	. . . .	$:\mathrm{Exp}\,ig_2\varphi_2:$ with $:\mathrm{Exp}\,ig_2\varphi_2:$	or
		$:\mathrm{Exp}-ig_2\varphi_2:$ with $:\mathrm{Exp}-ig_2\varphi_2:$	
f. $(p^2)^{\sigma-1}$	. . . .	$:\mathrm{Exp}\,ig_2\varphi_2:$ with $:\mathrm{Exp}-ig_2\varphi_2:$	
g. $(p^2)^{-\rho-1}$	. . . .	$:\mathrm{Exp}\,ig_1\varphi_1:$ with $:\mathrm{Exp}\,ig_1\varphi_1:$	or
		$:\mathrm{Exp}-ig_1\varphi_1:$ with $:\mathrm{Exp}-ig_1\varphi_1:$	
h. $(p^2)^{\rho-1}$	. . . .	$:\mathrm{Exp}\,ig_1\varphi_1:$ with $:\mathrm{Exp}-ig_1\varphi_1:$	

At this point we could introduce a graphical notation to represent the above propagators, but this is not essential. In any case it is rapidly seen that a regularization is necessary to avoid infrared divergences. To keep changes at a minimum only progators associated with the φ_2 field will be modified (recall that L_{Int} does not depend on φ_1). Because of charge conjugation the vertices of a graph G can be separeted into the following two sets. To the first set, V_1, belong the vertices which are connected to the external vertices of G by fermionic lines. The other set, V_2, contains the remaining vertices of G. The fermionic lines connecting the vertices in V_2 form, therefore closed loops. The regularization that we will employ can be now described:

1. If a exponentiated propagator links a vertex of V_1 with a vertex of V_2, we make the replacement

$$\mathrm{Exp}(\alpha\, D_F(x-y)) \quad \to \quad \mathrm{Exp}(\alpha\, \Delta_F(x-y, m^2)) \qquad (III.4)$$

where $\Delta_F(x, m^2)$ is the free propagator of mass m. The modification does not change the ultraviolet behaviour whereas at large distance we have

$$\mathrm{Exp}(\alpha \Delta_F(x)) \xrightarrow{r=\sqrt{x^2}\to\infty} 1 \qquad (III.5)$$

2. Otherwise, if both ends of a line are vertices in V_1 (or in V_2) then the momentum space propagator is changed as

$$(p^2)^{\alpha} \longrightarrow (p^2+m^2)^{\alpha} \qquad (III.6)$$

This regularization is not equivalent to (III.4). Indeed, the Fourier transform of the r.h.s. of (III.6) is not an exponential of a massive propagator but the function

$$\frac{2^{\alpha+2}\pi}{\Gamma(-\alpha)} (m^2)^{\alpha+1} \frac{K_{\alpha+1}(mr)}{r^{\alpha+1}} \qquad (III.7)$$

where $K_{\alpha+1}(mr)$ is a modified Bessel function. We observe that the substitution (III.6) gives a better large distance behaviour than (III.4), namely, if $r \to \infty$ then (III.7) tends to zero. The forthcoming dicussion will clarify the reasons for adopting two kinds of regulators instead of only one.

Returning to the study of the ultraviolet behaviour of the regulated Feynman integrands, we recall the definition of a generalized vertex. This is any subgraph obtained by deleting some of the vertices (and all lines meeting at these vertices) of the original graph. Only proper (1PI) generalized vertices can generate counterterms (7). Consider therefore a proper generalized vertex γ. We want to calculate the degree of superficial divergence of γ.

Let then p_1, p_2, ℓ_1 and ℓ_2 be the number of vertices of γ associated with the fields $:\mathrm{Exp}-2ig_2\varphi_2:\bar\psi_{01}\psi_{01}$, $:\mathrm{Exp}\,2ig_2\varphi_2:\bar\psi_{02}\psi_{02}$, $:\mathrm{Exp}(ig_1\varphi_1 - ig_2\varphi_2):\psi_{01}$ and $:\mathrm{Exp}(-ig_1\varphi_1 + ig_2\varphi_2):\psi_{02}$, respectively. Similarly let $\bar\ell_1$ and $\bar\ell_2$ indicate the number of vertices associated with $:\mathrm{Exp}(-ig_1\varphi_1 - ig_2\varphi_2):\bar\psi_{01}$ and $:\mathrm{Exp}(-ig_1\varphi_1 + ig_2\varphi_2):\bar\psi_{02}$. With this notation, the degree of superficial divergence of γ will be

$$\delta(\gamma) = 2 - p - \frac{3}{2}\ell - \frac{N_F}{2} + \frac{g_1^2}{4\pi}(\ell - f) + \frac{g_2^2}{4\pi}(\ell + 4p - h) \qquad (III.8)$$

where $\ell = \ell_1 + \ell_2 + \bar\ell_1 + \bar\ell_2$, $p = p_1 + p_2$

N_F = # of external fermionic lines

$$f = (\ell_1 + \ell_2 - \bar\ell_1 - \bar\ell_2)^2 \qquad (III.9)$$

$$h = (2p_1 - 2p_2 + \ell_1 + \bar\ell_1 - \ell_2 - \bar\ell_2)^2 \qquad (III.10)$$

IV. Ultraviolet Analysis

We first consider the diagrams of the unperturbed theory for which $N_F = p = f = h = 0$. We then have

$$\delta(\gamma) = 2 - \left(\frac{3}{2} - (\sigma+\rho)\right)\ell = 2 - \left(\frac{3}{2} - \frac{4\sigma^2}{1+4\sigma}\right)\ell \qquad (IV.1)$$

From this formula we see that the n point Green functions are well defined for

$$-\sqrt{3} + 1 < \frac{g_1^2}{\pi} < \sqrt{3} + 1 \qquad (IV.2)$$

Outside this interval the dimension of ψ_r becomes greater than one. In the repulsive region, the point $g_2^2 = \pi(\sqrt{3}+1)$ is above the point $\beta^2 = 8\pi$ where, as we will see, the model becomes unrenormalizable.

The a priori inexistence of the Green functions of the unperturbed model is without physical consequences. The cause is that the divergent parts are proportional to product of delta functions. The arguments of these delta functions are the coordinates differences of the external fields. Therefore, the divergent parts can be absorbed into a redefinition of the time ordering. By the same reason, the divergences of the full interacting theory associated with graphs with at least two external vertices (i.e. $\ell \geq 2$) can be eliminated by a mere redefinition of the time ordering. However these procedures can not be implemented by the addition of counterterms to the Lagrangian.

From the above observations, it is clear that we need to consider only the cases with $\ell < 2$. Within this constraint we examine each possibility:

1. $N_F=0, \ell=0$. Some illustrative graphs are depicted in fig. 1. Power counting, eq. (III.8), gives

$$\delta(\gamma) = 2 - (1 - 4\sigma)\, p$$

Thus, for $\sigma < 0$, δ is negative. For $0 < \sigma < 1/4$, which corresponds the interval $4\pi < \beta^2 < 8\pi$, δ is less than two. As $N_F = 0$ and also because of chiral symmetry $p_1 = p_2$. Therefore the reduced vertex $V(\gamma)$, obtained by contracting the graph γ to a point has no lines. Actually, for this to happen it is important to have a regularization like (III.4). Differently, had we uniformly employed the regularization (III.6), no cancellation of the external lines would occur. The divergence is partially removed by combining these graphs with the corresponding (disconnected) diagrams

coming from the denominator of the Gell-Mann Low formula. In fig. 2 we show a graph which becomes disconnected when the upper bubble is reduced to a point. For $\sigma < 1/8$ the divergence is only logarithmic and is entirely removed in this combination of graphs. For $1/8 < \sigma < 1/4$ the divergence becomes linear but, because of Lorentz covariance, no counterterm is necessary. For $\sigma > 1/4$ ($\beta^2 > 8\pi$), δ increases with p and the model becomes unrenormalizable. So, from now on we will restrict the analysis to $\sigma < 1/4$

2. $N_F = 1$, $\ell = 1$. Because of chiral symmetry and charge conservation $f = h = 1$ and therefore

$$\delta(\gamma) = 2 - (1 - 4\sigma)\, p - \left(\tfrac{3}{2} - (\sigma + \rho)\right) - \tfrac{1}{2} - \sigma - \rho = -(1 - 4\sigma)\, p < 0$$

3. $N_F = 2$. Since in this case ℓ must be even, we have to consider only the possibility $\ell = 0$. There are two subcases:

3.1. p is even. We have then $f = h = 0$. Thus

$$\delta(\gamma) = 1 - (1 - 4\sigma)\, p$$

For $\sigma < 0$ there is no divergence. For $\sigma > 0$, δ is less than one. However, as p is even the number of internal fermion lines of γ is odd. Therefore the divergence is absent if symmetric integration is employed.

3.2. p is odd. Here $f = 0$ but $h = 2$. Thus $\Delta = 1 + 4\sigma$ and

$$\delta(\gamma) = -(1 - 4\sigma)(p - 1) < 0$$

4. $N_F = 3$, $\ell = 1$. We have

4.1. p is even. Thus $f = h = 1$ so that $\Delta = 3/2 + \sigma + \rho$ and

$$\delta(\gamma) = 2 - \left[(1 - 4\sigma)\, p + \tfrac{3}{2} - (\sigma + \rho)\right] + \Delta = -1 - (1 - 4\sigma)\, p < 0$$

4.2. p is odd. Here, again, there are two subcases to consider:

4.2.1. f=h=1. We get $\Delta = 3/2 + \sigma + \rho$ and $\delta < 0$.

4.2.2. f=1 but h=9 (i.e. # of $(\psi_1 + \bar{\psi}_1)$ − # of $(\psi_2 + \bar{\psi}_2)$ =3) From this results $\Delta = 3/2 + 9\sigma + \rho$ and

$$\delta(\gamma) = -1 - [(1-4\sigma)p + 4\sigma] \leq -3$$

5. Now consider the case with $N_F > 4$ arbitrary and $\ell = 0$. We then have f=0 and, depending on γ, $p_1 - p_2$ can be equal to 0, 1,2,...,$N/2$. If $p_1 = p_2$ then the N_F external fields will consist of equal numbers of ψ_{01}, ψ_{02}, $\bar{\psi}_{01}$ and $\bar{\psi}_{02}$. In the other extreme case, i.e. when $p_1 - p_2 = N_F/2$, all the external fields will have the same index. Remember now that, because of charge conservation, the fermion lines can end or begin only at the vertices associated to the external fields. Let us treat a generic case in which the fermion lines link the external fields in the following way: x_1 paths connect x_1 ψ_{01} external fields to x_1 $\bar{\psi}_{01}$ external fields; x_2 connect x_2 ψ_{02} to x_2 $\bar{\psi}_{02}$; a_1 connect a_1 ψ_{01} to a_1 $\bar{\psi}_{02}$ and a_2 connect a_2 ψ_{02} to a_2 $\bar{\psi}_{01}$. Clearly, $x_1 + x_2 + a_1 + a_2 = N_F/2$. If this graph is divergent, a typical counterterm will be formed of a certain number of derivatives acting on a field monomial composed of the same ψ_0's as the external fields. The counterterm can be simplified using $\partial_0 \psi_2 = \partial_1 \psi_2$ (and $\partial_0 \psi_1 = -\partial_1 \psi_1$). Indeed, $\not{\partial}\psi = 0$ since $\not{\partial}$ cuts a fermion line leaving a result which contains $\mathrm{Exp}(\alpha \Delta_F(0))$ as a factor. This is zero if a convenient ultraviolet regularization (dimensional, for example) is employed. Because of this and Fermi statistics, there is a minimum number of derivatives which should be applied in order to get a non zero result. For example

$$\partial_0 \psi_{01} \partial_1 \psi_{01} \partial_0 \bar{\psi}_{01} \partial_1 \bar{\psi}_{01} \sim (\partial_0 \psi_{01})^2 (\partial_0 \bar{\psi}_{01})^2 = 0$$

It is not difficult to see that the minimum number of derivatives allowed is

$$D = \frac{1}{2}\left[(x_1 + a_1)(x_1 + a_1 - 1) + (x_2 + a_2)(x_2 + a_2 - 1) + (x_1 + a_2)(x_1 + a_2 - 1) + (x_2 + a_1)(x_2 + a_1 - 1)\right] \leq x_1^2 + x_2^2 - N_F/2$$

On the other hand, the degree of superficial divergence is

$$\delta(\gamma) = 2 - (1-4\sigma)p - N_F/2 - 4\sigma(x_1 - x_2)^2$$

For $\sigma > 0$, δ is negative. Also if $-(\sqrt{3} - 1)\frac{1}{4} < \sigma < 0$ then $D > \delta$ and the divergence will be cancelled.

The case with N_F arbitrary and $\ell = 1$ can be analysed analologously giving the same result.

This concludes our discussion of the ultraviolet behaviour of the Green functions. Summing up, we have shown that for $4\pi(2 - \sqrt{3}) < \beta^2 < 8\pi$ the only possible divergences are volume divergences which, nonetheless, cancel in the Gell-Mann Low formula

Now it is time to justify the use of the two regulators (III.4) and (III.6). The form of the regulator (III.4) enforces the cancellation of "vacuum bubble" diagrams, as explained (case 1). Since we want to keep ψ_0 massless then, due to (III.5), we need also the regulator (III.6) to hold infrared divergences away.

A similar discussion can be done for the construction of normal products of the bilinears, $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$, which are very important for the boson formulation of the model[2]. The graphs contributing to $\langle T\,\mathcal{O}(x)\psi(x_1)\ldots\psi(x_N)\bar\psi(y_1)\ldots\bar\psi(y_N)\rangle$ with $\mathcal{O}$ equal either to $\bar\psi\psi$ or to $\bar\psi\gamma^5\psi$) have a special vertex $V_{\mathcal{O}}$, associated to $\mathcal{O}$. However, since this vertex has the same structure as those coming from the interaction Lagrangian, the power counting will still be the same as before. Therefore, for $4\pi(2-\sqrt{3})<\beta^2<8\pi$ the only new divergences correspond to subgraphs which contain $V_{\mathcal{O}}$, have $p_1=p_2$ and have no external fermion lines. They are of the type 1, discussed previously. From the remarks there, it is clear that the divergent parts, which appear only for $4\pi<\beta^2<8\pi$, can be identified with contributions to the vacuum expectation value of $\mathcal{O}$. Now, because of charge conjugation (or parity) this vacuum expectation value is zero if $\mathcal{O}=\bar\psi\gamma^5\psi$. So we get the results that

$$N[\bar\psi\psi] = :\bar\psi\psi: - \langle :\bar\psi\psi: \rangle \tag{IV.3}$$

$$N[\bar\psi\gamma^5\psi] = :\bar\psi\gamma^5\psi: \tag{IV.4}$$

are well defined operators for $\beta^2<8\pi$ (we stress that the $\langle :\bar\psi\psi: \rangle$ in (IV.3) is necessary only for $4\pi<\beta^2<8\pi$ where it is divergent). This agree with Swieca's conjecture on composite operators of the sine-Gordon model.[5]

V. Current Conservation And Equations Of Motion

For the massive model we can still define a current analogous to (II.12)

$$g^\mu = \frac{1}{k}\left(g_1\,\partial^\mu \varphi_1 + g_2\,\tilde\partial^\mu \varphi_2 \right) \tag{V.1}$$

Indeed, this current is obviously conserved and satisfies ($Z_T = \psi_T(x_2)\ldots\psi_T(x_N)\bar\psi_T(y_1)\ldots\bar\psi_T(y_N)$)

$$\begin{aligned}
&\langle T\, g^\mu(x')\,\psi_T(x)\,Z_T\,L_{Int}(z_1)\ldots L_{Int}(z_q)\rangle = \Big\{ \frac{g_1^2}{k}\,\partial^\mu\Big[D_F(x'-x) + \sum_{i=2}^{N} D_F(x'-x_i) \\
&- \sum_{i=1}^{N} D_F(x'-y_i)\Big] + \frac{g_2^2}{k}\,\tilde\partial^\mu\Big[\gamma^5_x\,\Delta_F(x'-x) + \sum_{i=2}^{N}\gamma^5_{x_i}\,\Delta_F(x'-x_i) + \sum_{i=1}^{N}\gamma^5_{y_i}\Delta_F(x'-y_i)\Big] \\
&+ \frac{2g_2^2}{k}\,\tilde\partial^\mu \sum_{i=1}^{q}\gamma^5_{z_i}\,\Delta_F(x'-z_i)\Big\}\,\langle T\,\psi_T(x)\,Z_T\,L_{Int}(z_1)\ldots L_{Int}(z_q)\rangle
\end{aligned} \tag{V.2}$$

which shows explicitly the absence of further divergences. However, if we want to define the product $g^\mu(x)\psi(x)$ we should let $x' \to x$. In this situation additional divergences can appear. To see that in detail we have to consider two possibilities: 1. If x and x' are linked by just one line (propagator) we get graphs of the type shown in fig.3a. These divergences are not dangerous since they can be eliminated by Wick ordering. 2. If any path linking x to x' consists of more than one propagator we obtain graphs as that in fig.3b. Due to (V.2), the graph will contain a line associated to $\partial^\mu D_F(x-w)$ or to $\partial^\mu\Delta_F(x-w)$. This factor can be imagined as coming from the differentiation of an exponentiated propagator. In any case, the graph will be more singular, because of the additional momenta factor. Instead of giving an unmotivated definition for its finite part, we first examine the field equations where such product occurs. We have

$$\begin{aligned}
i\not\partial\langle T\psi(x)Z\rangle &= k\langle T :\gamma_\mu g^\mu\psi:(x)\,Z\rangle + M\big(\mathrm{Exp}\,2g_2^2\Delta_F(0)\big)_R\langle T\bar\psi(x)Z\rangle\; + \\
&+ i\sum_{i=1}^{N}(-1)^{i+N}\,\delta(x-y_i)\big[\mathrm{Exp}\,(g_1^2 D_F(0) + g_2^2\Delta_F(0))\big]_R\langle T\,Z_{y_i}\rangle
\end{aligned} \tag{V.3}$$

where Z is equal to Z_T with ψ_T replaced by ψ and Z_{y_i} is the same as Z with $\bar\psi(y_i)$ omitted. The indice R is to indicate that the quantity in parenthesis is infrared regulated as in (III.5) (or (III.4)). Note that, because of the factor $\mathrm{Exp}(2g_2^2\Delta_F(0))$, the second term on the r.h.s. of (V.3) is absent if $\sigma<0$. Moreover, for $\sigma > 0$ this term is divergent and should be used to compensate a corresponding divergence in the first term. At $\sigma = 0$ (V.3) becomes the Dirac equation for a free massive spinor field ψ.

The derivation of (V.3) is standard: In momentum space the graphs contributing to the left hand side of (V.3) have the structure shown in fig.4. Writing $\not p = \not p + \not K - \not K$ we get two terms. In the first of these two terms the $\not p + \not K$ factor is used to cancel a fermion propagator. This produces the second (if the cancelled propagator linked x to an interaction vertex) and the third (if the cancelled propagator linked x to an external vertex terms in the right hand side of (V.3). The remaining term, on the other hand, is easily recognized as a contribution to $\langle T{:}\gamma_\mu \partial^\mu \psi{:}(x)$ $Z\rangle$.

It is now clear that an useful definition of the finite part of the product of the current with the field is

$$\langle T\, N[\gamma^\mu \partial_\mu \psi](x)\, Z \rangle = \langle T{:}\gamma_\mu \partial^\mu \psi{:}(x)\, Z\rangle + \frac{M}{k}\left[\mathrm{Exp}(2 g_2^2 \Delta_F(0)) - 1\right]_R \langle T\psi(x) Z\rangle$$
$$+ \frac{i}{k}\sum_{i=1}^{N} (-1)^{i+N}\left[\mathrm{Exp}(g_1^2 D_F(0) + g_2^2 \Delta_F(0)) - 1\right]_R \langle T Z_{y_i} \rangle \qquad \text{(V.4)}$$

With this definition, the field equation takes the usual form

$$(i\not\partial - M)\langle T\psi(x) Z\rangle = k\langle T N[\gamma_\mu \partial^\mu \psi](x) Z\rangle + i\sum_{i=1}^{N} (-1)^{i+N} \delta(x-y_i)\langle T Z_{y_i}\rangle \qquad \text{(V.5)}$$

VI. Concluding Remarks

In this study of mass perturbation in the Thirring model we have verified that the Green functions are well defined for $4\pi(2-\sqrt{3}) < \beta^2 < 8\pi$. In this interval the only divergences are those associated with vacuum bubbles which cancel in the Gell-Mann

Low formula. For $\beta^2 > 8\pi$ the theory is not renormalizable: The degree of superficial divergence increases without bound with the order of perturbation and our methods are no longer applicable. Besides that, at $\beta^2 = 8\pi$ the propagator associated with a line linking two interaction vertices develops a non integrable singularity.

We have also shown that the mass operator can be made finite in the interval $4\pi(2-\sqrt{3}) < \beta^2 < 8\pi$ by subtracting its vacuum expectaction value besides the usual Wick ordering.

To avoid infrared divergences, it was necessary to introduce auxiliary mass regulators. The elimination of these regulators requires in principle an infinite resummation of the perturbative series. A possible way to accomplish that could be by writing the interaction as

$$N[\bar\psi_T \psi_T] = {:}[\mathrm{Exp}(2 i g_2 \gamma^5 \varphi_2) - 1]{:}\,{:}\bar\psi_0 \psi_0{:} + {:}\bar\psi_0 \psi_0{:} \qquad \text{(V.6)}$$

and then transferring the last term to the unperturbed Lagrangian. This would provide a mass to the free fermion propagator and possibly would eliminate the infrared divergences. But more graphs will have to be examined and they could generate additional ultraviolet divergences. The outcome of this analysis depend on the particular value of β^2. For $4\pi < \beta^2 < 8\pi$ the result is satisfactory since there is only one divergent graph, shown in fig.5. Such divergence can be compensated by adding a counterterm cte $\cos(2 g_2 \varphi_2)$ to the Lagrangian. The arbitrariness in the finite part can be fixed by imposing a definite value for the mass of the φ_2 field.

Amazingly, the same procedure does not work for $\beta^2 < 4\pi$. It happens that, in this region σ is negative, which favors the appearance of new divergent graphs. This is illustrated by the graph of fig.6 which contains a subgraph divergent for $\sigma < -\frac{1}{2N}$ (the associated counterterm will be a cosine of a higher harmonic of $2 g_2 \varphi_2$). We could say that, in this region, the net effect of the resummation is to replace infrared by ultraviolet divergences. A different resummation procedure, evading this situation would be highly desirable.

Acknowledgements. This work is part of a doctoral thesis by A.J.S. Partial support from CNPq (Conselho Nacional de Pesquisas),Brasil, is acknowledged.

Figure Captions

Fig 1 - Divergent graphs without exernal fermion lines. Solid and wavy lines represent fermion and exponentiated propagators, respectively. The + (or -) sign at the vertices indicates the corresponding sign of the exponentiated field.

Fig 2 - The lines connecting the vertices 3 to 1 and 2 (and 4 to 1 and 2) cancel, when the bubble is contracted to a point.

Fig 3 - Graphs contributing to $\lim_{x \to x'} g^\mu(x)\psi(x')$. (a) corresponds to the situation where x and x' were linked just by the indicated wavy line. Any other possibility produces graphs like (b).

Fig 4 - Graphical structure of the l.h.s. of (V.3)

Fig 5 - The only divergent graph in the region $4\pi < \beta^2 < 8\pi$, after the resummation (V.6). The vertex with the cross corresponds to the additional interaction $:\bar\psi_0\psi_0:$ coming from the resummation (V.6

Fig 6 - All vertices of the above graph have exponentiated fields (for simplicity, exponentiated propagators are not explicitly shown). The generalized subgraph made with the vertices on the fermionic loop has a degree of divergence increasing with N, if $\sigma < 0$.

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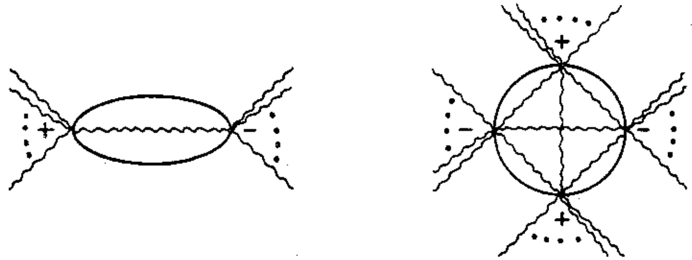

Fig. 1

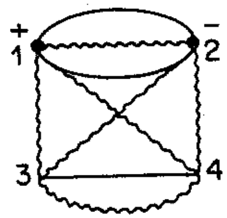

Fig. 2

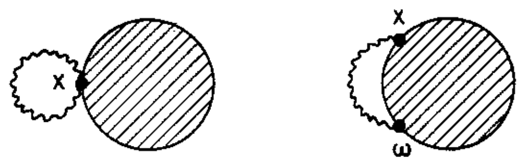

Fig. 3a

Fig. 3b

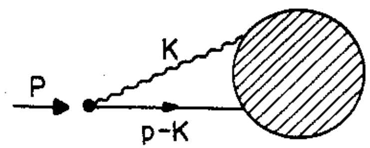

Fig. 4

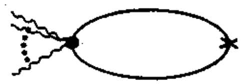

Fig. 5

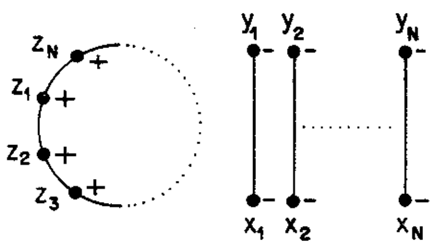

Fig. 6