UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-601

THE UNITARITY DEFECT OF THE S-MATRIX AND STATISTICAL MULTISTEP DIRECT NUCLEAR PROCESSES

by

M.S. Hussein
Instituto de Física, Universidade de São Paulo

Setembro/1986

THE UNITARITY DEFECT OF THE S-MATRIX AND STATISTICAL MULTISTEP DIRECT NUCLEAR PROCESSES

M.S. Hussein

Instituto de Física, Universidade de São Paulo
C.P. 20516, 01498 São Paulo, SP, Brazil

ABSTRACT

A relation is derived which connects the unitarity defect function $S^{\dagger}S-1$ with the imaginary part of the absorptive potential responsible for the nuclear scattering. The concept of angle-dependent reaction cross-section is introduced for the purpose. A similar relation is also obtained for the equivalent quantity $S^{\dagger}-S^{-1}$. Several applications to nuclear scattering are made, and possible relevance of our unitarity defect relation to statistical coupled channels theories of preequilibrium reactions is pointed out and discussed.

*Supported in part by the CNPq.

August/1986

I. INTRODUCTION

The description of multichannel reaction processes via equivalent one-channel theories has been a common procedure in several branches of nuclear physics. The vehicle through which this is accomplished is the energy-dependent complex optical potential

$$U_i(E) = P_i U P_i + P_i U Q_i (E^{(+)} - Q_i H Q_i)^{-1} Q_i U P_i \qquad (1)$$

$$P_i = |\psi_i\rangle\langle\psi_i| \quad , \quad Q_i = 1 - P_i \qquad (2)$$

where P_i projects onto the channel i. The operator $U_i(E)$ is explicitly energy-dependent, complex, and non-local because of the coupling to the other channels projected by Q_i. Given $U_i(E)$, the diagonal transition amplitude and correspondingly the elastic S-matrix element in channel i are obtained from the solution of a standard one-channel Lippmann-Schwinger equation

$$T_i(E) = P_i T(E) P_i = U_i(E) + U_i(E) G_o^{(+)}(E) T_i(E) \qquad (3)$$

or, equivalently the optical Schrödinger equation

$$(E - P_i H_o P_i - U_i(E)) |\Psi_E^{(+)}\rangle = 0 \qquad (4)$$

In Eq. (3), $G_o^{(+)}(E)$ represents the Green function that describes free propagation in the channel i and H_o is the diagonal piece of the total Hamiltonian. Besides $|\psi_E^{(+)}\rangle$, three other wave functions can be introduced. These are the incoming solution to (4), $\langle\psi_E^{(-)}|$ and the dual wave functions $|\tilde{\psi}_E^{(-)}\rangle$ and $\langle\tilde{\psi}_E^{(+)}|$ defined such that

$$\langle\tilde{\Psi}^{(+)}_{\vec{k}'}|\Psi^{(+)}_{\vec{k}}\rangle = (2\pi)^3\delta(\vec{k}'-\vec{k}) \tag{5}$$

$$\langle\Psi^{(-)}_{\vec{k}'}|\tilde{\Psi}^{(-)}_{\vec{k}}\rangle = (2\pi)^3\delta(\vec{k}'-\vec{k}) \tag{6}$$

Owing to the energy-dependence and non-hermiticity of $U_i(E)$ the wave function $|\psi^{(\pm)}_{\vec{k}}\rangle$ is not orthonormal; $\langle\psi^{(\pm)}_{\vec{k}'}|\psi^{(\pm)}_{\vec{k}}\rangle \neq (2\pi)^3\delta(\vec{k}'-\vec{k})$. The S-matrix is obtained as usual from the inner product

$$\langle\vec{k}'|S|\vec{k}\rangle = \langle\Psi^{(-)}_{\vec{k}'}|\Psi^{(+)}_{\vec{k}}\rangle \tag{7}$$

As long as the energy dependence of U_i is maintained, no simple relations between the wave functions $|\psi^{(\pm)}\rangle$ and their corresponding dual states $|\tilde{\psi}^{(\pm)}\rangle$ exist[1]. In fact, to obtain $|\tilde{\psi}^{(+)}\rangle$ from $|\psi^{(+)}\rangle$, one has to solve a rather intricate integral equation. One has[1]

$$\langle\vec{p}|\Psi^{(+)}_{\vec{k}}\rangle = (2\pi)^3\delta(\vec{p}-\vec{k}) + \Phi^{(+)}_{\vec{k}}(\vec{p}) \tag{8}$$

$$\langle\tilde{\Psi}^{(+)}_{\vec{q}}|\vec{p}\rangle = (2\pi)^3\delta(\vec{p}-\vec{q}) - \tilde{\Phi}^{(+)}_{\vec{q}}(\vec{p}) \tag{9}$$

$$\tilde{\Phi}^{(+)}_{\vec{q}}(\vec{k}) = \Phi^{(+)}_{\vec{k}}(\vec{q}) - \int\frac{d\vec{p}}{(2\pi)^3}\,\Phi^{(+)}_{\vec{k}}(\vec{q})\,\tilde{\Phi}^{(+)}_{\vec{q}}(\vec{p}) \tag{10}$$

In low-energy nuclear reactions, the optical potential $U_i(E)$ varies slowly with energy and is usually taken to be locally energy-independent. Then the dual states $\tilde{\psi}^{(+)}$ ($\tilde{\psi}^{(-)}$) are given simply by the solution of the Schrödinger equation with $U(U^\dagger)$ replaced by $U^\dagger(U)$. Explicitly, we have the following L-S equations satisfied by $|\psi^{(+)}\rangle$, $|\tilde{\psi}^{(+)}\rangle$, $|\psi^{(-)}\rangle$ and $|\tilde{\psi}^{(-)}\rangle$

$$|\Psi^{(+)}_{\vec{k}}\rangle = |\vec{k}\rangle + G_o^{(+)}(E_k)\,U\,|\Psi^{(+)}_{\vec{k}}\rangle \equiv \Omega^{(+)}_{\vec{k}}|\vec{k}\rangle \tag{11}$$

$$|\tilde{\Psi}^{(+)}_{\vec{k}}\rangle = |\vec{k}\rangle + G_o^{(+)}(\epsilon_k)\,U^\dagger\,|\Psi^{(+)}_{\vec{k}}\rangle \equiv \tilde{\Omega}^{(+)}_{\vec{k}}|\vec{k}\rangle \tag{12}$$

$$|\Psi^{(-)}_{\vec{k}}\rangle = |\vec{k}\rangle + G_o^{(-)}(\epsilon_k)\,U^\dagger|\Psi^{(-)}_{\vec{k}}\rangle \equiv \Omega^{(-)}_{\vec{k}}|\vec{k}\rangle \tag{13}$$

$$|\tilde{\Psi}^{(-)}_{\vec{k}}\rangle = |\vec{k}\rangle + G_o^{(-)}(\epsilon_k)\,U\,|\tilde{\Psi}^{(-)}_{\vec{k}}\rangle \equiv \tilde{\Omega}^{(-)}_{\vec{k}}|\vec{k}\rangle \tag{14}$$

Of course $|\psi^{(+)}_{\vec{k}}\rangle$ and $|\psi^{(-)}_{\vec{k}}\rangle$ are the physical states. In elastic scattering as well as in DWBA calculations, these are the only wave functions needed. They, of course, satisfy the usual time-reversal relation

$$\langle -\vec{p} \,|\, \Psi^{(+)}_{\vec{k}} \rangle^{*} = \langle \vec{p} \,|\, \Psi^{(-)}_{\vec{k}} \rangle \qquad (15)$$

A similar relation holds between $|\tilde{\Psi}^{(+)}\rangle$ and $|\tilde{\Psi}^{(-)}\rangle$.

In several recent development in reaction theory, the dual states have attained a marked importance. Specifically, the multi-step processes encountered in preequilibrium reactions, which require the use of the spectral representation of the distorted Green function and thus products of the type $|\tilde{\Psi}^{(+)}\rangle\langle\Psi^{(+)}|$, have been greatly debated . A recent paper by Feshbach[2] addressed several features of the dual states and their connection with the inverse S-matrix. He applied his findings to Σ-nucleus reactions[3]. Our purpose in the present paper is to extend Feshbach's results and derive relations between the unitarity defect of the S-matrix, exemplified by $S^{\dagger}S-1$ and/or $S^{\dagger}-S^{-1}$, and the underlying absorpbtive potential U . Several interesting properties of the dual states are also pointed out. Numerical evaluation of $S^{\dagger}S-1$ is also made.

The plan of the paper is as follows. In Section II we derive our expressions for $S^{\dagger}S-1$ and $S^{\dagger}-S^{-1}$ and discuss several transformation properties of the scattering and dual states. In Section III we apply our result to several nuclear scattering cases. The angle-dependent total reaction cross-section which appears in $S^{\dagger}S-1$ is calculated both for heavy ion (strong absorption) and nucleon (weak absorption) elastic

scattering situations. In Section IV we discuss the relevance of our findings to the statistical multi-step direct reaction theory of Feshbach, Kerman and Koonin[4], and finally, in Section V, we present our discussion and conclusions.

II. TRANSFORMATION PROPERTIES OF THE DUAL STATES AND THE UNITARITY DEFECT OPERATORS $S^{\dagger}-S^{-1}$ AND $S^{\dagger}S-1$

In this Section we discuss the transformation properties of the dual states, defined by Eqs. (12) and (14). In particular, we calculate inner products of the type $\langle\Psi^{(+)}_{\vec{k}'}|\Psi^{(+)}_{\vec{k}}\rangle$, $\langle\Psi^{(-)}_{\vec{k}'}|\Psi^{(-)}_{\vec{k}}\rangle$, $\langle\tilde{\Psi}^{(+)}_{\vec{k}'}|\tilde{\Psi}^{(-)}_{\vec{k}}\rangle$, etc.. This is accomplished by first deriving relations between $|\Psi^{(\pm)}_{\vec{k}}\rangle$ and its dual state $|\tilde{\Psi}^{(\pm)}_{\vec{k}}\rangle$. From these relations we calculate the above mentioned inner products, as well as $S^{\dagger}-S^{-1}$ and $S^{\dagger}S=1$.

From Eqs. (11)-(14), we can write the following equivalent representations of the plane-wave solution

$$|\vec{k}\rangle = \Omega^{(+)-1}_{\vec{k}} \,|\, \Psi^{(+)}_{\vec{k}}\rangle \qquad (16a)$$

$$|\vec{k}\rangle = \tilde{\Omega}^{(+)-1}_{\vec{k}} \,|\, \tilde{\Psi}^{(+)}_{\vec{k}}\rangle \qquad (16b)$$

$$|\vec{k}\rangle = \Omega^{(-)-1}_{\vec{k}} \,|\, \Psi^{(-)}_{\vec{k}}\rangle \qquad (16c)$$

$$|\vec{k}\rangle = \tilde{\Omega}^{(-)-1}_{\vec{k}} \,|\, \tilde{\Psi}^{(-)}_{\vec{k}}\rangle \qquad (16d)$$

We now use Eq. (16b) in Eq. (11) to obtain the relation between $|\psi_{\vec{k}}^{(+)}\rangle$ and $|\tilde{\psi}_{\vec{k}}^{(+)}\rangle$.

$$|\Psi_{\vec{k}}^{(+)}\rangle = \tilde{\Omega}_{\vec{k}}^{(+)^{-1}} |\tilde{\Psi}_{\vec{k}}^{(+)}\rangle + G_0^{(+)}(\epsilon_k)\, U\, |\Psi_{\vec{k}}^{(+)}\rangle \qquad (17)$$

From the following identity

$$G_0^{(+)}(\epsilon_k) = \tilde{\Omega}_{\vec{k}}^{(+)^{-1}} \tilde{G}^{(+)}(\epsilon_k) \qquad (18)$$

where $\tilde{G}^{(+)}(E_k) = (E_k - H_0 - U^\dagger + i\eta)^{-1}$, we have

$$|\Psi_k^{(+)}\rangle = \tilde{\Omega}_{\vec{k}}^{(+)^{-1}} |\tilde{\Psi}_{\vec{k}}^{(+)}\rangle + \tilde{\Omega}_{\vec{k}}^{(+)^{-1}} \tilde{G}^{(+)}(E_k)\, U\, |\Psi_{\vec{k}}^{(+)}\rangle \qquad (19)$$

or

$$|\Psi_{\vec{k}}^{(+)}\rangle = \tilde{\Omega}_{\vec{k}}^{(+)^{-1}} |\tilde{\Psi}_{\vec{k}}^{(+)}\rangle + \tilde{\Omega}_{\vec{k}}^{(+)^{-1}} \tilde{G}^{(+)}(\epsilon_k)\, U^\dagger\, |\Psi_{\vec{k}}^{(+)}\rangle + \tilde{\Omega}_{\vec{k}}^{(+)} \tilde{G}^{(+)}(\epsilon_k)\,(U - U^\dagger)\, |\Psi_{\vec{k}}^{(+)}\rangle \qquad (20)$$

Upon multiplying Eq. (20) from the left by $\tilde{\Omega}_{\vec{k}}^{(+)}$, and using the definition of $\tilde{\Omega}_{\vec{k}}^{(+)}$ (see Eq. (12)), we can rewrite Eq. (20) in the following form

$$|\Psi_{\vec{k}}^{(+)}\rangle = |\tilde{\Psi}_{\vec{k}}^{(+)}\rangle + \tilde{G}^{(+)}(\epsilon_k)\,(U - U^\dagger)\, |\Psi_{\vec{k}}^{(+)}\rangle \qquad (21)$$

Eq. (21) clearly exhibit the connection between $|\psi_{\vec{k}}^{(+)}\rangle$ and

$|\tilde{\psi}_k^{(+)}\rangle$, through the absorptive piece of the optical potential. Clearly we can rewrite (21) as

$$|\Psi_{\vec{k}}^{(+)}\rangle = |\tilde{\Psi}_{\vec{k}}^{(+)}\rangle + G^{(+)}(\epsilon_k)\,(U - U^\dagger)\, |\tilde{\Psi}_{\vec{k}}^{(+)}\rangle \qquad (22)$$

Similar relations can be derived for the other physical wave function

$$|\Psi_{\vec{k}}^{(-)}\rangle = |\tilde{\Psi}_{\vec{k}}^{(-)}\rangle + G^{(-)}(\epsilon_k)\,(U - U^\dagger)\, |\tilde{\Psi}_{\vec{k}}^{(-)}\rangle \qquad (23)$$

$$= |\tilde{\Psi}_{\vec{k}}^{(-)}\rangle + \tilde{G}^{(-)}(\epsilon_k)\,(U - U^\dagger)\, |\Psi_{\vec{k}}^{(-)}\rangle \qquad (24)$$

where $G^{(-)}(E_k) = (E_k - H_0 - U^\dagger - i\epsilon)^{-1}$ and $\tilde{G}^{(-)}(E_k) = (E_k - H_0 - U - i\epsilon)^{-1}$.

It is now a simple matter to obtain expressions for the different inner products,

$$\langle \Psi_{\vec{k}'}^{(+)} | \Psi_{\vec{k}}^{(+)} \rangle = (2\pi)^3 \delta(\vec{k} - \vec{k}') + \frac{\langle \Psi_{\vec{k}'}^{(+)} | (U - U^\dagger) | \Psi_{\vec{k}}^{(+)} \rangle}{E_k - E_{k'} + i\varepsilon} \qquad (25)$$

$$\langle \Psi_{\vec{k}'}^{(-)} | \Psi_{\vec{k}}^{(-)} \rangle = (2\pi)^3 \delta(\vec{k} - \vec{k}') + \frac{\langle \Psi_{\vec{k}'}^{(-)} | (U - U^\dagger) | \Psi_{\vec{k}}^{(-)} \rangle}{E_k - E_{k'} - i\varepsilon} \qquad (26)$$

where we have used the normalization conditions, Eqs. (5) and (6). From time reversal invariance we know that the matrix elements $\langle \psi_{\vec{k}}^{(+)} | U - U^\dagger | \psi_k^{(+)} \rangle$ and $\langle \psi_{\vec{k}'}^{(-)} | U - U^\dagger | \psi_{\vec{k}}^{(-)} \rangle$ are equal.

Then we can immediately calculate the difference between the above two inner products

$$\langle \Psi^{(+)}_{\vec{k}'} | \Psi^{(+)}_{\vec{k}} \rangle - \langle \Psi^{(-)}_{\vec{k}'} | \Psi^{(-)}_{\vec{k}} \rangle = -2\pi i \delta(E_k - E_{k'}) \times \tag{27}$$

$$\times \langle \Psi^{(+)}_{\vec{k}'} | (U - U^{\dagger}) | \Psi^{(+)}_{\vec{k}} \rangle$$

$$= -4\pi \delta(E_k - E_{k'}) \langle \Psi^{(+)}_{\vec{k}'} | | \mathrm{Im} U | | \Psi^{(+)}_{\vec{k}} \rangle \tag{28}$$

Since the matrix element $\langle \psi^{(+)}_{\vec{k}} | | \mathrm{Im} U | | \psi^{(+)}_{\vec{k}} \rangle$ is just $\frac{E}{k} \sigma_R$ with σ_R being the total reaction cross section, we can introduce an angle-dependent total reaction cross-section, $\sigma_R(\theta)$, given by

$$\sigma_R(\theta) = \frac{k}{E} \langle \Psi^{(+)}_{\vec{k}'} | | \mathrm{Im} U | | \Psi^{(+)}_{\vec{k}} \rangle \tag{29}$$

through which Eq. (28) can be rewritten as

$$\langle \Psi^{(+)}_{\vec{k}'} | \Psi^{(+)}_{\vec{k}} \rangle - \langle \Psi^{(-)}_{\vec{k}'} | \Psi^{(-)}_{\vec{k}} \rangle = -4\pi \frac{E_k}{k} \delta(E_{k'} - E_k) \sigma_R(\theta) \tag{30}$$

Since, from time reversal, $\langle \bar{\psi}^{(+)}_{\vec{k}'} | \bar{\psi}^{(+)}_{\vec{k}} \rangle = \langle \psi^{(+)}_{\vec{k}'} | \psi^{(+)}_{\vec{k}} \rangle$ and $\langle \psi^{(-)}_{\vec{k}'} | \bar{\psi}^{(-)}_{\vec{k}} \rangle = \langle \psi^{(-)}_{\vec{k}'} | \psi^{(-)}_{\vec{k}} \rangle$, a relation similar to Eq. (30) should hold for the difference $\langle \bar{\psi}^{(+)}_{\vec{k}} | \bar{\psi}^{(+)}_{\vec{k}} \rangle - \langle \bar{\psi}^{(-)}_{\vec{k}} | \bar{\psi}^{(-)}_{\vec{k}} \rangle$.

In a way, Eqs. (25) and (26) represent an extension of the Bell-Steinberger relations[5], involving resonance reactions, to scattering from a complex optical potential. In the former case, the relations involve the norm of the resonance

wave function on the one hand and the ratio between twice the matrix element of the imaginary part of the resonance hamiltonian taken with respect to the resonance wave function, and the width of the resonance, on the other hand. Of course no relation similar to Eq. (30) exists for the resonance case, since there is only one relevant boundary condition.

We turn now to the S-matrix defined through the follwoing matrix element, Eq. (7)

$$\langle \vec{k}' | S | \vec{k} \rangle = \langle \Psi^{(-)}_{\vec{k}'} | \Psi^{(+)}_{\vec{k}} \rangle \tag{31}$$

The inverse of S is immediately found from the completeness relation involving $|\psi^{(+)}\rangle \langle \bar{\psi}^{(+)}|$, and Eq. (6), namely

$$\int \frac{d\vec{k}''}{(2\pi)^3} \langle \vec{k}' | S | \vec{k}'' \rangle \langle \vec{k}'' | S^{-1} | \vec{k} \rangle = (2\pi)^3 \delta(\vec{k} - \vec{k}')$$

$$\int \frac{d\vec{k}''}{(2\pi)^3} \langle \Psi^{(-)}_{\vec{k}'} | \Psi^{(+)}_{\vec{k}''} \rangle \langle \tilde{\Psi}^{(+)}_{\vec{k}''} | \tilde{\Psi}^{(-)}_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k} - \vec{k}') \tag{32}$$

Therefore we have, as does Feshbach ,

$$\langle \vec{k}' | S^{-1} | \vec{k} \rangle = \langle \tilde{\Psi}^{(+)}_{\vec{k}'} | \tilde{\Psi}^{(-)}_{\vec{k}} \rangle \tag{33}$$

We can now relate S^{-1} to $S^{\dagger}$, using our wave function transformation relations 21-24. We calculate first $\langle \bar{\psi}^{(-)}_{\vec{k}} | \bar{\psi}^{(+)}_{\vec{k}} \rangle$. From Eq. (21) we can write

$$\langle \tilde{\Psi}^{(-)}_{\vec{k}'} | \tilde{\Psi}^{(+)}_{\vec{k}} \rangle = \langle \tilde{\Psi}^{(-)}_{\vec{k}'} | \Psi^{(+)}_{\vec{k}} \rangle - \frac{\langle \tilde{\Psi}^{(-)}_{\vec{k}'} | (U - U^\dagger) | \Psi^{(+)}_{\vec{k}} \rangle}{E_{\vec{k}} - E_{\vec{k}'} + i\varepsilon} \quad (34)$$

and from Eq. (23), we have

$$\langle \tilde{\Psi}^{(-)}_{\vec{k}'} | \Psi^{(+)}_{\vec{k}} \rangle = \langle \Psi^{(-)}_{\vec{k}'} | \Psi^{(+)}_{\vec{k}} \rangle + \frac{\langle \tilde{\Psi}^{(-)}_{\vec{k}'} | (U - U^\dagger) | \Psi^{(+)}_{\vec{k}} \rangle}{E_{\vec{k}'} - E_{\vec{k}} + i\varepsilon} \quad (35)$$

Using Eq. (35) in Eq. (34), we have the desired relation

$$\langle \tilde{\Psi}^{(-)}_{\vec{k}'} | \tilde{\Psi}^{(+)}_{\vec{k}} \rangle = \langle \Psi^{(-)}_{\vec{k}'} | \Psi^{(+)}_{\vec{k}} \rangle + 2\pi i \delta(\varepsilon_k - \varepsilon_{k'}) \langle \tilde{\Psi}^{(-)}_{\vec{k}'} | (U - U^\dagger) | \Psi^{(+)}_{\vec{k}} \rangle \quad (36)$$

Taking the complex conjugate of (36) interchanging $\vec{k}$ and $\vec{k}'$ and using the time reversal property of the matrix element $\langle \tilde{\Psi}^{(-)}_{\vec{k}'} | (U - U^\dagger) | \Psi^{(+)}_{\vec{k}} \rangle$, we have finally

$$\langle \vec{k}' | S^{-1} | \vec{k} \rangle = \langle \vec{k}' | S^\dagger | \vec{k} \rangle + 2\pi i \delta(\varepsilon_k - \varepsilon_{k'}) \langle \tilde{\Psi}^{(+)}_{\vec{k}'} | (U - U^\dagger) | \Psi^{(+)}_{\vec{k}} \rangle \quad (37)$$

Eq. (37) can be further simplified by inserting the complete set $\int | \tilde{\Psi}^{(-)}_{\vec{k}''} \rangle \langle \Psi^{(-)}_{\vec{k}''} | \frac{d^3\vec{k}''}{(2\pi)^3}$ inside the matrix and using the definition of S^{-1}, Eq. (33) again. The following integral equation is finally obtained

$$\langle \vec{k}' | S^{-1} | \vec{k} \rangle = \langle \vec{k}' | S^\dagger | \vec{k} \rangle + 2\pi i \delta(\varepsilon_k - \varepsilon_{k'}) \times \int \frac{d\vec{k}''}{(2\pi)^3} \langle \vec{k}' | S^{-1} | \vec{k}'' \rangle \langle \Psi^{(-)}_{\vec{k}''} | (U - U^\dagger) | \Psi^{(-)}_{\vec{k}} \rangle \quad (38)$$

Since all matrix elements are completely on-the-energy shell, the integral in the above Eq. (38) reduces to an integral our solid angle only. Removing an energy delta function from all the operators above, we can then write

$$S^{-1}_{\vec{k},\vec{k}'} = S^\dagger_{\vec{k},\vec{k}'} + \frac{k^2}{(2\pi)^2} \int d\Omega_{\hat{k}''} S^{-1}_{\vec{k},\vec{k}''} \sigma_R(\vec{k}'',\vec{k}') \quad (39)$$

where σ_R is the angle-dependent total cross section introduced earlier, Eq. (29). Eq. (39) is simply a generalization of what is known as Moldauer's sum rule[6] in resonance reactions, to the scattering from complex optical potential. When expressed in partial waves, Eq. (39) reduces to the following relation

$$S_\ell^{-1} = S_\ell^* + S_\ell^{-1} \sigma_\ell^{(\ell)} \quad (40)$$

or the well-known result

$$\sigma_R^{(\ell)} = 1 - |S_\ell|^2 \quad (41)$$

The importance of Eq. (39), however, is in cases where a

partial-wave representation is not advantageous such as in the statistical multi-step direct reaction threory of Feshbach, Kerman and Koonin[4]. Since the operator S^{-1} is required in the final result of FKK , owing to the use of the spectral representation of the Green function, a transformation of the type given in Eq. (39) could be of great use. This point will be fully discussed in Section IV.

An immediate consequence of Eq. (38) is a closed expression for the unitarity defect function $SS^{\dagger}-1$. Upon multiplying Eq. (38) from the left by S and integrating over intermediate plane wave states, we obtain the following

$$\langle \vec{k}' | (SS^{\dagger}-1) | \vec{k} \rangle = -4\pi \delta(E_k - E_{k'}) \langle \Psi^{(-)}_{\vec{k}'} | (U-U^{\dagger}) | \Psi^{(-)}_{\vec{k}} \rangle \quad (42)$$

$$= -\left(\frac{4\pi E_k}{k}\right) \delta(E_k - E_{k'}) \, \sigma_R(\theta) \quad (43)$$

Equations (39) and (43) constitutes the principal results of this section. In particular, Eq. (43) exhibits the deviation from unitarity of the elastic S-matrix, exemplified by $S^{\dagger}S \neq 1$, through a deviation of the momentum transfer $\vec{k}' - \vec{k} \equiv \vec{q}$, from zero value. The degree of this deviation is measured by the angle-dependent total reaction cross-section, $\sigma_R(\vec{k}.\vec{k}')$. Clearly this function is maximum in the forward direction and oscillatory vs. θ . Its value at $\theta = 0$ is just the total reaction cross-section σ_R . In the next section we investigate in details the behaviour of $\sigma_R(\vec{k}.\vec{k}')$ for several nuclear scattering systems.

III. THE ANGLE-DEPENDENT TOTAL REACTION CROSS SECTION $\sigma_R(\vec{k}.\vec{k}')$

In this section we analyse and calculate the angle-dependent total reaction cross-section $\sigma_R(\vec{k}.\vec{k}')$, which was shown in the previous section to represent the degree of unitarity defect of the optical S-matrix. We consider both nucleon- and nucleus-nucleus elastic scattering. We also test the sensitivity of σ_R to different optical potentials.

We take first the heavy-ion elastic scattering case, as it represents the least model-dependent case owing to the strong absorption that dominates the scattering process. We first write the partial-wave-sum of $\sigma_R(\vec{k}'.\vec{k})$,

$$\sigma_R(\vec{k}'.\vec{k}) = \frac{k}{E_k} \langle \Psi^{(+)}_{\vec{k}'} | \, | \mathrm{Im} U | \, | \Psi^{(+)}_{\vec{k}} \rangle \quad (44)$$

$$= \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \, T_\ell \, P_\ell(\cos\theta) \quad (45)$$

where T_ℓ is the ℓ^{th} partial wave transmission coefficient given by

$$T_\ell = 1 - |S_\ell|^2 \quad (46)$$

$$= \frac{8\mu k}{\hbar^2} \int_0^{\infty} dr \, |\psi_\ell(r)|^2 \, |\mathrm{Im} U(r)| \quad (47)$$

In the above $\psi_\ell(r)$ is the radial wave function in the ℓ^{th} partial wave. Owing to the strong absorption exhibited by the heavy-ion system, one may accurately characterize T_ℓ by the limiting or grazing angular momentum, ℓ_g, that specifies the boundary in angular momentum space between the strong and the weak absorption regions. For simplicity we write for T_ℓ, the sharp cut-off form

$$T_\ell = \Theta(\ell_g - \ell) \tag{48}$$

where Θ is the step function. Consequently we have for $\sigma_R(\vec{k}'.\vec{k})$ the following

$$\sigma_R(\vec{k}'.\vec{k}) = \frac{\pi}{k^2}\sum_{\ell=0}^{\ell_g}(2\ell+1)P_\ell(\cos\theta) \tag{49}$$

Many years ago Blair[7] derived an exact closed expression for the elastic scattering amplitude of neutral particle scattering treated within the sharp cut off model being discussed here. Since $\sigma_R(\vec{k}'.\vec{k})$ has a similar partial wave sum structure as the elastic amplitude, we shall follow in below the steps followed by Blair[7]. We first remind the reader of the following recursion relation among the Legendre polynomial

$$(2\ell+1)P_\ell = P'_{\ell+1} - P'_{\ell-1} \tag{50}$$

where

$$P'_\ell(\cos\theta) = \frac{d}{d(\cos\theta)}P_\ell(\cos\theta)$$

Using the fact that $P'_0 = 0$ and $P'_1 = 1$, we have immediately the desired closed expression of Eq. (49)

$$\sigma_R(\vec{k}.\vec{k}') = \frac{\pi}{k^2}\left(P'_{\ell_g+1} + P'_{\ell_g}\right) \tag{51}$$

Eq. (51) can be further simplified with the help of the following relation between the Legendre polynomials and ordinary Bessel functions of zero order,

$$P_\ell(\cos\theta) \simeq J_0[(\ell+\tfrac{1}{2})\theta] \tag{52}$$

Since $\ell_g >> 1$ in heavy-ion scattering, we may place the sum $P'_{\ell_g+1} + P'_{\ell_g} \approx 2P'_{\ell_g+\frac{1}{2}}$. We thus find, with the help of Eq. (52) the following, Fraunhofer diffraction type expression for σ_R

$$\sigma_R(\vec{k}.\vec{k}') = -\frac{\pi}{k^2}\frac{\ell_g+1}{\theta}\left[-\frac{d}{d[(\ell_g+1)\theta]}J_0[(\ell_g+1)\theta]\right]$$

$$= \frac{\pi}{k^2}(\ell_g+1)^2\,\frac{J_1[(\ell_g+1)\theta]}{[\ell_g+1]\theta}$$

$$= \sigma_R(\vec{k}^2)\,\frac{J_1[(\ell_g+1)\theta]}{[(\ell_g+1)\theta]} \tag{53}$$

where $\sigma_R(k^2) \equiv \frac{\pi}{k^2}(\ell_g+1)^2 = \frac{\pi}{k^2}\sum_{\ell=0}^{\ell_g}(2\ell+1)$ is the total reaction cross-section evaluated in the sharp cut-off limit.

Clearly at $\theta = 0$, $\sigma_R(\vec{k}'.\vec{k})$, Eq. (53), is just $\sigma_R(\vec{k}^2) = \frac{\pi}{k^2}(\ell_g+1)^2$, as it should. At small angles (ℓ_g+1) may be identified with the momentum transfer times the interaction (strong absorption) radius,

$$(\ell_g+1)\theta \simeq kR\theta = (k\theta)R \cong \left(2k \operatorname{sen}\frac{\theta}{2}\right)\theta = qR \qquad (54)$$

In (54), we have ignored the Coulomb barrier (high energy approximation). This restriction, however, can be relaxed by retaining the $(\ell_g+1)\theta$ form and taking for ℓ_g+1 the Barrier-modified form

$$(\ell_g+1) \simeq kR\left[1 - \frac{V_B}{E}\right]^{1/2} \qquad (55)$$

where V_B is the height of the Coulomb barrier. Going back to the high energy representation of $(\ell_g+1)\theta$, we have as a consequence, the following

$$\sigma_R(\vec{k}'.\vec{k}) \simeq \sigma_R(q) = \sigma_R\left[\frac{J_1[qR]}{qR}\right] \qquad (56)$$

The above equation shows that $\sigma_R(\vec{k}'.\vec{k})$ is roughly a function of the momentum transfer q, on which it depends through the

Fraunhofer diffraction amplitude-like $(J_1(qR)/qR)$. Clearly at certain values of qR, $\sigma_R(q)$ becomes negative rendering $\langle\vec{k}'|SS^\dagger|\vec{k}\rangle$ larger than "unity" and thus violating unitarity locally. By unity here we mean a delta function in $\vec{q}$. On the average, however, $\langle\vec{k}'|SS^\dagger|\vec{k}\rangle$ comes out "smaller" than a delta function since the interaction is absorptive ($\mathrm{Im}U < 0$). We also note from Eq. (56) or (53) that at large enough values of qR or ℓ_g, $\sigma_R(\vec{k}'.\vec{k})$ becomes practically zero, indicating that $\langle\vec{k}'|S^\dagger S|\vec{k}\rangle$ becomes on the average a delta function $(2\pi)^3\delta(\vec{k}-\vec{k}')$. Thus, absorption, in the sense of $S^\dagger S$, is not felt at large values of ℓ_g.

In order to exhibit the above features of $\sigma_R(\vec{k}'.\vec{k})$ more quantitatively, we show in Fig. (2) the result of our calculation using a smooth cut-off model for T_ℓ, namely

$$T_\ell = \left[1 + \exp\left(\frac{\ell_g - \ell}{\Delta}\right)\right]^{-1} \qquad (57)$$

For the parameters ℓ_g and Δ we have taken the values 25 and 2, respectively. For simplicity we have set $k^2 = \pi$. We see from the Figure that the period of angle oscillation of σ_R is roughly $\frac{2\pi}{(\ell_g+1)}$, as can be easily evaluated from Eq. (53). The envelope of the oscillation drops faster than that of Eqs. (53) and (51) owing to the non-zero value of Δ, in Eq. (53). An approximate analytical expression for $\sigma_R(\vec{k}.\vec{k}')$ appropriate when $\Delta \neq 0$, can be worked out easily. One finds[7)]

$$\sigma_R(\vec{k}'\cdot\vec{k}) \simeq \sigma_R \frac{J_1[(\ell_g+1)\theta]}{(\ell_g+1)\theta} F_1(\Delta\theta) \qquad (58)$$

where the smooth cut-off damping function $F_1(\Delta\theta)$ is given by (when a Woods-Saxon form of T_ℓ is used, Eq. (57))

$$F_1(\Delta\theta) = \frac{(\pi\Delta\theta)}{\sinh(\pi\Delta\theta)} \qquad (59)$$

At large enough values of Δ, $F_1(\Delta\theta)$ behaves like $\frac{1}{2}\pi\Delta\theta\exp[-\pi\Delta\theta]$, which clearly becomes exceedingly small if Δ is appreciable.

As a consequence, Eq. (58) serves to distinguish between cases of strong absorption, (Δ small and ℓ_g large), intermediate absorption (Δ large and ℓ_g large, and weak absorption (Δ large and ℓ_g small). The words strong, intermediate and weak we are using here to characterize absorption, refer to the angular range over which $\sigma_R(\vec{k}'.\vec{k})$ is appreciable on the average. For example in very heavy-ion systems at above barrier energies one has $\ell_g >> 1$, $\frac{\Delta}{\ell_g} << 1$ with Δ of the order of several units of $\hbar$ and thus, according to our criterion above, the HI system is a strong-absorption system. In contrast, light-ion reactions, are invariably characterized by small ℓ_g (implying large value of $[(\ell_g+1)\theta]^{-1}$, Eq. (58)) and not so small Δ (compared to ℓ_g). Thus $\sigma_R(\vec{k}',\vec{k})$ is appreciable over a rather wide angular ($\theta = \cos^{-1}(\vec{k}'.\vec{k})$) region. This

we call, a typical weak-absorption system.

We turn now to this latter case of weak-absorption scattering system, exemplified here by neutron-nucleus scattering at low energies. To be specific we consider $n + {}^{16}O$ at E_n = 20 MeV. For the purpose, we use three different optical potentials, proposed to fit neutron-nucleus elastic scattering at $0 < E_n < 100$ MeV. These are the Beccheti-Greenles[8] Hodgson-Wilmore[9], and Rapaport[10] optical potentials. In figure 2 we present the corresponding transmission coefficients. In all three cases, T_ℓ exhibits features quite different from that of Eq. (57). There is a very conspicuous maximum at $\ell=3$, owing, partly, to the importance of surface absorption in these potentials. However, the two quantities which characterize T_ℓ namely ℓ_g and Δ, are still the important ones in determining $\sigma_R(\theta)$. This we show in Fig. (3) for the three optical potentials cited above.

We note that the 180^o to 0^o anisotropy of $\sigma_R \equiv \frac{\sigma(180^o)}{\sigma(0^o)}$, is four times as large as that of the heavy-ion case, Fig. 1. In this latter case this anisotropy can be evaluated exactly in the sharp cut-off model, Eq. (49). We find

$$R = \frac{(-)^{\ell_g}}{(\ell_g+1)} \qquad (60)$$

Clearly, for very heavy ion scattering systems characterized by

$\ell_g >>$, R becomes very small. This means that in these situations $\sigma_R(\vec{k}'.\vec{k})$ will have large values only in the small angle region. Therefore, on the average (ignoring the -oscillations) heavy-ion systems exhibit weak absorption features at large angles. It would seem therefore that a reasonable approximation to $\sigma_R(\vec{k}'.\vec{k})$ valid for large ℓ_g, would be

$$\sigma_R(\vec{k}.\vec{k}') \simeq \sigma_R \, \delta(\hat{k}-\hat{k}') \tag{61}$$

However, Eq. (61) would be valid if the function multiplying $\sigma_R(\vec{k}.\vec{k}')$ in the integrand of Eq. (39), namely $S^{-1}_{\hat{k}\hat{k}''}$ is a smooth function of $\hat{k}.\hat{k}''$. This is not the case here since $S^{-1}_{\hat{k}\hat{k}''}$ is expected to be highly oscillatory. In fact, in the very strong absorption-sharp-cut off limit discussed earlier and exemplified by $T_\ell = \Theta(\ell_g-\ell)$, $S^{-1}_{\hat{k}\hat{k}''}$ is obviously infinite. However, as we have seen from the numerical examples shown in Fig. 3, the strong absorption limit is never realized in light-ion scattering, except in the anomalously strongly absorved partial waves discussed by Kawai et al.[11].

We consider a schematic model for absorption which mocks up the realistic situation of Fig. (2), but which makes possible the explicit construction of $\sigma_R(\hat{k}'.\hat{k})$ and correspondingly $S^{-1}_{\hat{k}\hat{k}'}$. This model assumes the following form for the partial-wave S-coefficient S_ℓ,

$$S_\ell = x\,\Theta(\ell_g-\ell) + \Theta(\ell-\ell_g) \tag{62}$$

namely in the interval $\ell < \ell_g$, $S_\ell = x \neq 0$ and for $\ell > \ell_g |S_\ell| = 1$. Then simple calculation following the one used in the evaluation of σ_R in the strong absorption (x=0) sharp-cut-off approximation leads immediately to (no real nuclear or Coulomb phases are assumed present, for simplicity)

$$S^{\dagger}_{\hat{k}\hat{k}'} = S_{\hat{k}\hat{k}'} = \delta(\hat{k}-\hat{k}') + (x-1)\left[P'_{\ell_g+1}(\hat{k}.\hat{k}') + P'_{\ell_g}(\hat{k}.\hat{k}')\right] \tag{63}$$

$$\sigma_R(\hat{k}.\hat{k}') = (1-x^2)\frac{\pi}{k^2}\left[P'_{\ell_g+1}(\hat{k}.\hat{k}') + P'_{\ell_g}(\hat{k}.\hat{k}')\right] \tag{64}$$

$$S^{-1}_{\hat{k}\hat{k}'} = \delta(\hat{k}-\hat{k}') = \left(\frac{1-x}{x}\right)\left[P'_{\ell_g+1}(\hat{k}.\hat{k}') + P'_{\ell_g}(\hat{k}.\hat{k}')\right] \tag{65}$$

Using the approximation relating P_ℓ to the Bessel function, Eq. (52), we may now write (for small θ)

$$S^{\dagger}_{\hat{k}\hat{k}'} = S_{\hat{k}\hat{k}'} = \delta(\hat{k}-\hat{k}') + (x-1)(\ell_g+1)^2\,\frac{J_1[(\ell_g+1)\theta]}{(\ell_g+1)\theta} \tag{66}$$

$$\sigma_R(\hat{k}.\hat{k}') = (1-x^2)\,\frac{\pi}{k^2}(\ell_g+1)^2\,\frac{J_1[(\ell_g+1)\theta]}{(\ell_g+1)\theta} \tag{67}$$

$$S^{-1}_{\hat{k}\hat{k}'} = \delta(\hat{k}-\hat{k}') + \left(\frac{1-x}{x}\right)(\ell_g+1)^2\,\frac{J_1[(\ell_g+1)\theta]}{(\ell_g+1)\theta} \tag{68}$$

The above equations serve to interpolate, in the sharp-cut-off limit, between weak ($x \sim 1$) and strong ($x \ll 1$) absorption limits. It should be a simple matter to prove that Eq. (69) is a solution of Eq. (39), given Eq. (66) and (68). The modifications which are needed to make the above formulae valid in the smooth cut-off case, have already been discussed earlier in connection with Eq. (58). In fact it suffices to multiply Eqs. (66)-(68) by the damping factor $F_1(\Delta\theta)$, Eq. (59).

Finally to approximately take into account the effect of the real nuclear and Coulomb phase shifts in S^{-1}, so far neglected in our discussion, one may resort to the near-far decomposition of J_1, namely the running wave decomposition, and add to the exponents of each of the two branches appropriate angle-independent phases. Frahn and Venter[12] have devised methods to evaluate analytically $S_{k\hat{k}}$ in cases where the real nuclear phase shift varies smoothly with ℓ. The same methods can be easily extended to the evaluation of $S^{-1}_{\hat{k}\hat{k}'}$.

Armed with the above discussion and findings, we turn now to an application to a particular nuclear reaction namely preequilibrium processes. This we present in the following Section.

IV. STATISTICAL MULTISTEP DIRECT REACTION THEORIES AND THE UNITARITY DEFECT OF THE OPTICAL S-MATRIX

Pre-equilibrium nuclear reactions have been invariably described in the past several years, as examples of statistical multistep direct processes. Both heavy-ion deep inelastic reactions and light-ion induced pre-equilibrium reactions are treated as such. In the theory of Feshbach, Kerman and Koonin (FKK), the statistical multistep direct part of the emission is calculated as a convolution of DWBA-type one-step cross sections. Three basic approximations were employed by FKK to arrive at their cross-section form. The first two approximations are: 1) the use of the on-energy-shell approximation for the intermediate channel green functions and, 2) reduce the number of intermediate momenta (directions) which are integrated over to N-1, where N is the number of steps. These two approximations, applied in conjunction with statistical ensemble averaging, result in a SMD cross section which is a convolution of one-step cross-sections. E.g. for the two-step contribution is

$$\frac{d\sigma^{(2)}(E_f,E_i,E_1)}{d\Omega_f} = \frac{k_1^2}{16\pi^2}\int d\hat{k}_1\ \sigma^{(1)}(\vec{k}_f,\vec{k}_1)\cdot\tilde{\sigma}^{(1)}(\vec{k}_1,\vec{k}_i) \qquad (69)$$

and the three-step cross section is

$$\frac{d\sigma^{(3)}}{d\Omega_f}(E_f,E_i;E_1,E_2) = \frac{k_1^2}{16\pi^2}\frac{k_2^2}{16\pi^2}\int d\hat{k}_1\int d\hat{k}_2 \cdot$$

$$\cdot\,\sigma^{(1)}(\vec{k}_f,\vec{k}_2)\cdot\tilde{\sigma}^{(1)}(\vec{k}_2,\vec{k}_1)\cdot\tilde{\sigma}^{(1)}(\vec{k}_1,\vec{k}_i) \qquad (70)$$

In Eqs. (69) and (70), two kinds of one-step cross-sections appear inside the integral. These we call $\sigma^{(1)}$ and $\tilde{\sigma}^{(1)}$. As one may notice only the last step transition is described by $\sigma^{(1)}$. All other steps are described by $\tilde{\sigma}^{(1)}$. The difference between these cross sections is

$$\sigma^{(1)}(k_f,k_{n-1}) \propto |\langle\Psi^{(-)}_{\vec{k}_f}|\,v\,|\Psi^{(+)}_{\vec{k}_{n-1}}\rangle|^2 \qquad (71)$$

and

$$\tilde{\sigma}^{(1)}(\vec{k}_\ell,\vec{k}_m) \propto |\langle\tilde{\Psi}^{(+)}_{\vec{k}_\ell}|\,v\,|\Psi^{(+)}_{\vec{k}_m}\rangle|^2 \qquad (72)$$

and accordingly $\sigma^{(1)}$ is a genuine DWBA cross section, whereas $\tilde{\sigma}^{(1)}$ is an "anomalous" DWBA cross-section. In (71) and (72), v denotes the residual interaction.

In the actual numerical evaluation of the multistep cross-section done by Bonetti et al.[13], however $\tilde{\sigma}^{(1)}$ was **taken to be exactly** the same as $\sigma^{(1)}$, namely the difference

between $\langle\tilde{\Psi}^{(+)}|$ and $\langle\Psi^{(-)}|$ was ignored. This implies, according to our discussion in the previous section, and to Udagawa et al.[14], setting $S^{-1} = \mathbf{1}$, or more explicitly $S^{-1}(E_1,\hat{k}_1,\hat{k}_1') = \delta(\hat{k}_1-\hat{k}_1')$. This clearly means also $S = \mathbf{1}$, and thus no scattering at all in the intermediate stages. Said differently, such an approximation is analogous to approximating the wave functions $\langle\Psi^{(-)}_{\vec{k}}|$ and $\langle\tilde{\Psi}^{(+)}_{\vec{k}}|$ by plane waves!

Quite recently, however, Feshbach[15] brought up the question of an energy averaging to be performed on the multistep amplitude, which would, under strong absorption conditions, render $\langle\tilde{\Psi}^{(+)}| \approx \langle\Psi^{(-)}|$ and accordingly $\tilde{\sigma}^{(1)} \sim \sigma^{(1)}$. He argued that because of the strong absorption which characterizes S, resonance-like "states" (bound states in the continuum) would dominate S^{-1} and thus creating energy fluctuations. The energy average of this fluctuating multistep direct amplitude would then give rise to a cross-section of the type (69) and (70), composed of DWBA single-step cross-section. Namely $\langle\langle\tilde{\Psi}^{(+)}_{\vec{k}}|\rangle\rangle_{\Delta E} \approx \langle\Psi^{(-)}_{\vec{k}}|$. Following the same reasoning, one would also have $\langle|\tilde{\Psi}^{(-)}_{\vec{k}}\rangle\rangle_{\Delta E} = |\Psi^{(+)}_{\vec{k}}\rangle$, since $|\tilde{\Psi}^{(-)}_{\vec{k}}\rangle = \sum_{k'} S^{-1}_{\vec{k}\vec{k}'}\,|\Psi^{(+)}_{\vec{k}'}\rangle$ (see Eq. (33)) and $\langle S^{-1}\rangle = 1$. Consequently, we have

$$\begin{aligned}\langle\langle\vec{k}'|S^{-1}|\vec{k}\rangle\rangle_{\Delta E} &= \langle\langle\tilde{\Psi}^{(+)}_{\vec{k}'}|\tilde{\Psi}^{(-)}_{\vec{k}}\rangle\rangle_{\Delta E}\\ &= \langle\Psi^{(-)}_{\vec{k}'}|\Psi^{(+)}_{\vec{k}}\rangle\\ &= \langle\vec{k}'|S|\vec{k}\rangle = \delta(\vec{k}-\vec{k}')\end{aligned} \qquad (73)$$

Thus the consequence of energy averaging is the removal of all scattering in the intermediate stages. Whereas Feshbach claims that this is a consequence of the physics, Udagawa et al.[14] and also Hussein and Bonetti[16] consider it as a third approximation.

We shall not dwell here on the justification of either of the above claims. Rather we discuss below alternative means of deriving the statistical multistep cross section within the two approximations mentioned earlier in this section. For this purpose we shall use arguments based on our discussion in the previous section of $\sigma_R(\theta)$ and S^{-1}. In particular, because of the oscillatory nature of $S^{-1}_{\hat{k}',\hat{k}}$, it is conceivable that the cross section $\tilde{\sigma}^{(1)}$ comes out not quite different from $\sigma^{(1)}$ due to what we may call angle-self-averaging (ASA).

As we have seen in the previous section, a reasonable representation of $S^{-1}_{\hat{k}\hat{k}'}$, which exhibits clearly its -oscillation is the sharp-cut-off expression, Eq. (68)

$$S^{-1}_{\hat{k}\hat{k}'} = \left[\delta(\hat{k}-\hat{k}') + \left(\frac{1-x}{x}\right)(\ell_g+1)^2 \frac{J_1[(\ell_g+1)\theta]}{(\ell_g+1)\theta}\right] \qquad (74)$$

As discussed earlier, the modifications necessary for smooth cut-off absorption and refraction, can be easily incorporated in (74) following Frahn and Venter[12]. For our purpose below Eq. (74) suffices. With (74), the anomalous DWBA amplitude becomes

$$\langle \tilde{\psi}^{(+)}_{\vec{k}'} | v | \psi^{(+)}_{\vec{k}} \rangle = \langle \psi^{(-)}_{\vec{k}'} | v | \psi^{(+)}_{\vec{k}} \rangle + \left(\frac{1-x}{x}\right)(\ell_g+1)^2 \int d\hat{k}'' \cdot \frac{J_1[|\vec{k}'-\vec{k}''|R]}{|\vec{k}'-\vec{k}''|R} \langle \psi^{(-)}_{\vec{k}''} | v | \psi^{(+)}_{\vec{k}} \rangle \qquad (75)$$

where we have used Eq. (54) to rewrite the argument of the Bessel function. The Fraunhofer diffraction function $J_1(x)/x$ is depicted in Fig. (4) vs. $\theta \equiv \cos^{-1}(\hat{k}'.\hat{k}'')$. A possible reasonable approximation which can be used is to set $J_1(x)/x \approx \approx \frac{1}{2}\delta(\hat{k}''-\hat{k}')$. This would be O.K. if the DWBA amplitude oscillates much more rapidly than $J_1(x)/x$, within the angular interval of interest. Thus we set

$$\langle \tilde{\psi}^{(+)}_{\vec{k}'} | v | \psi^{(+)}_{\vec{k}} \rangle \simeq \langle \psi^{(-)}_{\vec{k}'} | v | \psi^{(+)}_{\vec{k}} \rangle \left[1 + \frac{1-x}{2x}(\ell_g+1)^2\right] \qquad (76)$$

$$\equiv \langle \psi^{(-)}_{\vec{k}'} | v' | \psi^{(+)}_{\vec{k}} \rangle \qquad (77)$$

Therefore the anomalous DWBA amplitude can be replaced by a normal DWBA amplitude, however, with a renormalized residual interaction.

Although the factor that multiplies v, namely

$$\frac{v'}{v} = \left[1 + \frac{1-x}{2x}(\ell_g+1)^2\right] \equiv \eta \qquad (78)$$

may acquire large values in heavy-ion reactions which are characterized by small x and large ℓ_g, its numerical value in nucleon - or light ion-induced reactions could be quite modest. From our calculation in Fig. (3), we find for $n + {}^{16}O$ at $E_n = 20$ MeV, $\eta \approx 12$. We should stress that this is an upper limit since taking into account the surface effect (smooth cut-off) and the real nuclear phases would effectively reduce the value of ℓ_g (since the function $F_1(\Delta\theta)$ damps $J_1(x)/x$). Accordingly one may very well find values for η smaller than, say, 2. The multistep cross section, then becomes, taking as an example the three-step contribution

$$\frac{d\sigma^{(3)}}{d\Omega_f}(E_f, E_i, E_1, E_2) = \frac{k_1^2}{16\pi^2}\,\frac{k_2^2}{16\pi^2}\,\eta^4 \cdot \int d\hat{k}_1 \int d\hat{k}_2 \cdot$$

$$\cdot\, \sigma^{(1)}(\vec{k}_f, \vec{k}_2) \cdot \sigma^{(1)}(\vec{k}_2, \vec{k}_1) \cdot \sigma^{(1)}(\vec{k}_1, \vec{k}_i) \qquad (79)$$

Thus, in order to proceed, one has to first perform a careful analysis of the factor η, considering realistic optical potentials. It is quite conceivable that the numerical values of η relevant for pre-equilibrium reactions dominated by MSD processes, are such as not to jeopardize the convergence of the FKK multistep series. The remaining part of the calculation, namely the evaluation of the convolution integrals, is exactly the same as the one performed by Bonetti et al.[13]. Of course

special case must be taken when confronted with cases of anomalously strong absorption of a particular partial wave of the kind discussed by Kawai et al.[11]. On the average, this effect would increase the value of η (reduce the value of x). Convergence problems may arise in such cases.

V. CONCLUSIONS AND DISCUSSION

In this paper we have derived transformation relations involving the scattering wave functions $|\psi_{\vec{k}}^{(+)}\rangle$ and $|\psi_{\vec{k}}^{(-)}\rangle$ and their respective dual states $|\tilde{\psi}_{k}^{(+)}\rangle$ and $|\tilde{\psi}_{\vec{k}}^{(-)}\rangle$. With the aid of these relations we were able to derive a new relation in scattering theory involving the conjugate of the S-matrix and its inverse. The basic quantity that enters in this relation, Eq. (39) is the angle dependent total reaction cross-section $\sigma_R(\hat{k}.\hat{k}')$. The properties of σ_R is then analysed in the context of nucleon-nucleus and nucleus-nucleus scattering. In light of these results, a rather complete discussion of the properties of S^{-1} in momentum space was presented. Closed expressions were obtained for $S^{-1}_{\hat{k}',\hat{k}}$ and $\sigma_R(\hat{k}.\hat{k}')$ in the sharp cut-off approximation.

The results for $S^{-1}_{\hat{k}',\hat{k}}$ were then utilized in the development of a slightly modified statistical multistep direct cross-section, which involves convolutions of single-step DWBA cross sections, just as in FKK, but with a renormalized residual interaction. It was pointed out that special care should be taken in cases of anomalously strongly absorbed partial waves. The renormalized residual interaction in such cases may become sufficiently large to render the multistep series slowly convergent.

It would be important to perform a careful analysis

of the renormalization parameter, η, through realistic optical model calculations, keeping in mind the eventual construction of a more realistic closed expression such as the sharp cut-off form, Eq. (74). The numerical investigation of our theory is in progress and will be reported later.

REFERENCES

1) M.S. Hussein and E.J. Moniz, Phys. Rev. 29C (1984) 2054.

2) H. Feshbach, Ann. Phys. (N.Y.) 164 (1985) 398.

3) H. Feshbach, Phys. Lett. (1986).

4) H. Feshbach, A.K. Kerman and S.E. Koonin, Ann. Phys. (N.Y.) 125 (1980) 429.

5) J.S. Bell and J. Steinberger in: Proc. Oxford Intern. Conf. on Elementary Particles, 1965, eds. T.R. Walsch et al. (Rutherford High Energy Laboratory, Chilton, Didcot, Berkshire, England, 1966).

6) P. Moldauer, Phys. Rev. Lett. (1965).

7) J. Blair, Lectures at the Tata Institute (1965).

8) F.D. Beccheti and G.W. Greenles, Phys. Rev. 182 (1969) 1190.

9) D. Wilmore and D.E. Hodgson, Nucl. Phys. 55 (1964) 673.

10) J. Rapaport et al., Nucl. Phys. A330 (1979) 15.

11) M. Kawai and Y. Iseri, Phys. Rev. C31 (1985) 400; Y. Iseri and M. Kawai, Fukuoka Preprint (1986).

12) W.E. Frahn and R.H. Venter, Ann. Phys. (N.Y.) 24 (1963) 243.

13) R. Bonetti, N. Camnasio and L. Colli-Milazzo, Phys. Rev. C24 (1981) 71.

14) T. Udagawa, T. Tamura and H. Lenske, Phys. Rev. C26 (1982) 379.

15) H. Feshbach, Ann. Phys. (N.Y.) (1984).

16) M.S. Hussein and R. Bonetti, Phys. Lett. 112B (1982) 189.

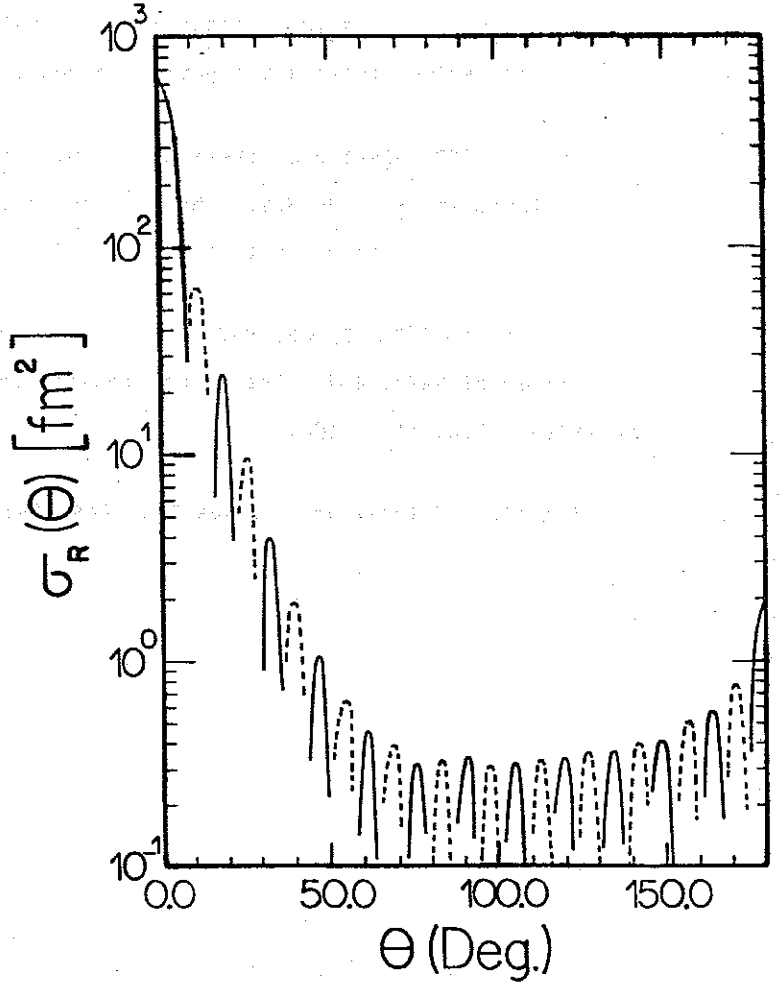
FIGURE CAPTIONS

Fig. 1 - A typical heavy-ion $\sigma_R(\theta)$, see text for details. The dashed regions represent negative values of $\sigma_R(\theta)$.

Fig. 2 - Typical $n + {}^{16}O$ elastic scattering transmission coefficients at E_n = 20 MeV. The results were obtained with the optical potentials of Refs. 6-7.

Fig. 3 - $\sigma_R(\theta)$ for $n + {}^{16}O$, $E_n = 20$ MeV, for the optical potentials of Refs. 6-7. The dashed regions represent negative values of $\sigma_R(\theta)$.

Fig. 4 - The function $F_1(x)/x$ vs. x, see Eqs. (66)-(68).


10^3
10^2
10^1
10^0
10^{-1}
$\sigma_R(\Theta)$ [fm^2]
0.0
50.0
100.0
150.0
Θ (Deg.)

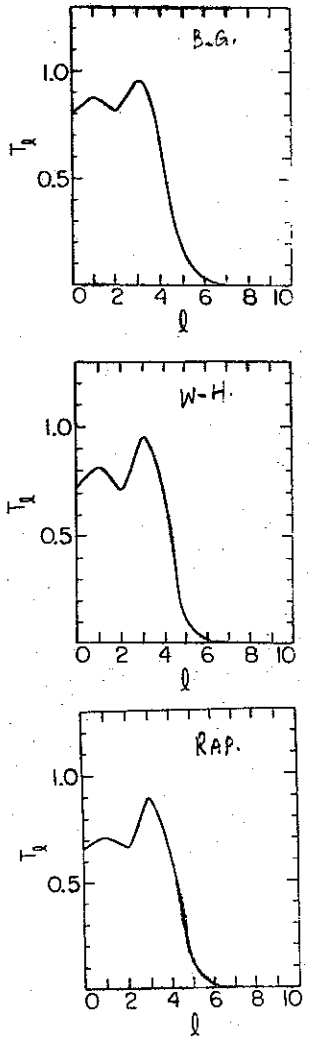

B.G.
W-H.
RAP.
1.0
0.5
T_ℓ
0 2 4 6 8 10
ℓ

Fig. 2

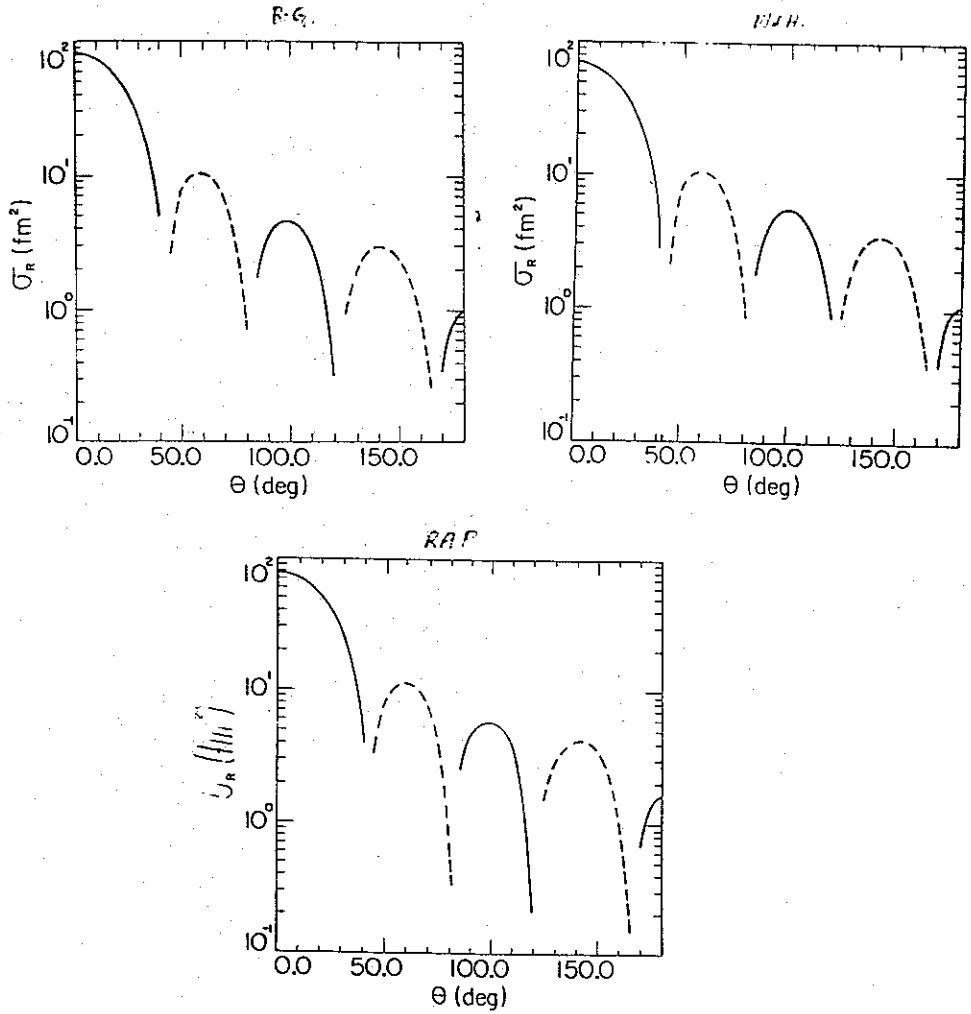

B-G.
σ_R (fm^2)
10^2
10^1
10^0
10^{-1}
0.0
50.0
100.0
150.0
θ (deg)

Fig. 3

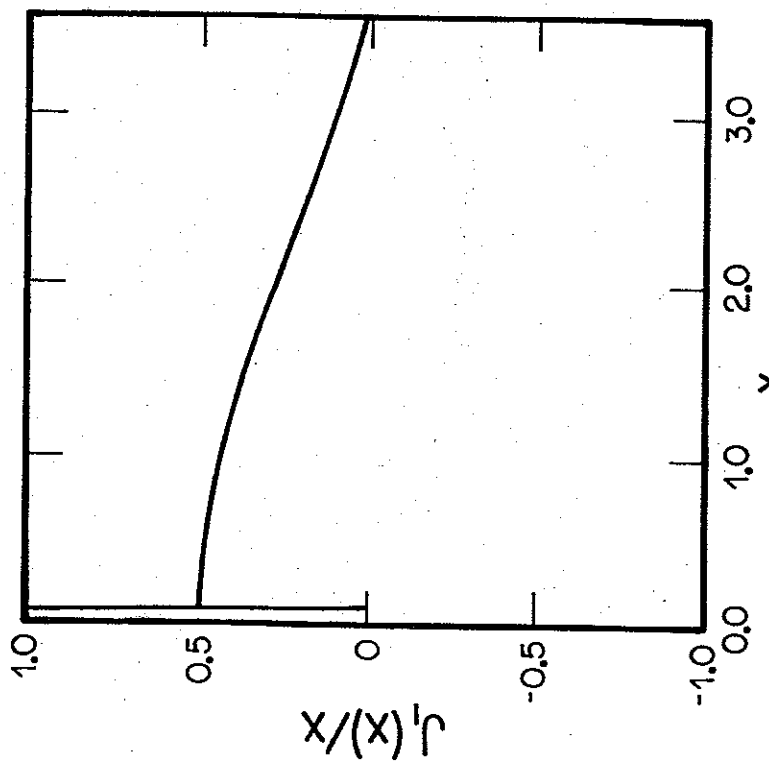

$J_1(X)/X$
1.0
0.5
0
-0.5
-1.0
0.0
1.0
2.0
3.0
X

Fig. 4