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THE UNITARITY DEFECT OF THE S-MATRIX AND THE UNDERLYING ABSORPTIVE POTENTIAL II: THE CASE OF ENERGY-DEPENDENT OPTICAL POTENTIALS

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THE UNITARITY DEFECT OF THE S-MATRIX AND THE UNDERLYING

ABSORPTIVE POTENTIAL II: THE CASE OF ENERGY-DEPENDENT

OPTICAL POTENTIALS*

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ABSTRACT

The unitarity defect relation involving $\langle \vec{k}'|S^{\dagger}S-1|\vec{k}\rangle$ on the one hand and the matrix element $\langle \psi_{\vec{k}}^{(+)}|U-U^{\dagger}|\psi_{\vec{k}}^{(+)}\rangle$ on the other, derived recently by Hussein,is extended here to energy-dependent interactions. This is accomplished through the introduction of two equivalent-energy-independent potentials connected with the outgoing and ingoing wave boundary conditions.

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I. INTRODUCTION

In a recent paper (in the following called I), we have discussed the connection between the unitarity defect function $\langle \vec{k}'|(S^{\dagger}S-1)|\vec{k}\rangle$ and the underlying absorptive interaction. Explicit formulae were derived under the assumption of energy-independent potentials. In particular, the following expression was obtained

$$\langle \vec{k}'|(S^{\dagger}S-1)|\vec{k}\rangle = \sigma_R(\vec{k}'\cdot\vec{k}) \qquad (1)$$

where $\sigma_R(\vec{k}'\cdot\vec{k})$ is an angle-dependent total reaction cross-section, given by

$$\sigma_R(\vec{k}'\cdot\vec{k}) = \frac{k}{E}\langle \psi_{\vec{k}'}^{(+)}|(U-U^{\dagger})|\psi_{\vec{k}}^{(+)}\rangle \qquad (2)$$

with $|\psi_{\vec{k}}^{(+)}\rangle$ being the exact outgoing wave function and U is the optical potential.

In the present short paper we extend the results of I to energy-dependent interactions. This is done through the introduction of two equivalent-energy-independent interactions connected with the outgoing and ingoing wave boundary conditions. The final results are obtained under the assumption of zero spin-orbit interaction. The paper is organized as follows. In Section II the major steps used in I for the

obtention of Eqs. (1) and (2) are reviewed. The extension to energy-dependent potentials is then presented in Section III. Finally, in Section IV, several concluding remarks are made.

II. THE UNITARITY DEFECT OF THE S-MATRIX AND ENERGY-INDEPENDENT OPTICAL POTENTIALS

In this section we present a short review of I. The basic starting point used in I is the Lippman-Schwinger equations for the physical states $|\psi_{\vec{k}}^{(+)}\rangle$ and $|\psi_{\vec{k}}^{(+)}\rangle$

$$|\Psi_{\vec{k}}^{(+)}\rangle = |\vec{k}\rangle + G_0^{(+)}(\varepsilon_k)\, U\, |\Psi_{\vec{k}}^{(+)}\rangle \equiv \Omega_{\vec{k}}^{(+)}\, |\vec{k}\rangle \qquad (3)$$

$$|\Psi_{\vec{k}}^{(-)}\rangle = |\vec{k}\rangle + G_0^{(-)}(\varepsilon_k)\, U^{\dagger}\, |\Psi_{\vec{k}}^{(-)}\rangle \equiv \Omega_{\vec{k}}^{(-)}\, |\vec{k}\rangle \qquad (4)$$

where U is the complex optical potential. From the orthogonality relations

$$\langle \widetilde{\Psi}_{\vec{k}'}^{(+)} | \Psi_{\vec{k}}^{(+)}\rangle = (2\pi)^3\, \delta(\vec{k}' - \vec{k}) \qquad (5)$$

$$\langle \Psi_{\vec{k}'}^{(-)} | \widetilde{\Psi}_{\vec{k}}^{(-)}\rangle = (2\pi)^3\, \delta(\vec{k}' - \vec{k}) \qquad (6)$$

and the completeness relations (we assume no bound states present)

$$\int \frac{d\vec{k}}{(2\pi)^3}\, |\Psi_{\vec{k}}^{(+)}\rangle \langle \widetilde{\Psi}_{\vec{k}}^{(+)}| = \mathbb{1} \qquad (7)$$

$$\int \frac{d\vec{k}}{(2\pi)^3}\, |\widetilde{\Psi}_{\vec{k}}^{(-)}\rangle \langle \Psi_{\vec{k}}^{(-)}| = \mathbb{1} \qquad (8)$$

we can obtain the L-S equations for the dual states $|\widetilde{\psi}_{\vec{k}}^{(+)}\rangle$ and $|\widetilde{\psi}_{\vec{k}}^{(-)}\rangle$. If the optical potential is taken to be energy-independent, as was the case in I, the L-S equations for the dual states $|\psi_{\vec{k}}^{(+)}\rangle$ and $|\psi_{\vec{k}}^{(-)}\rangle$ come out simply as (3) and (4) with $U \to U^{\dagger}$ and $U^{\dagger} \to U$, respectively. Taking U to be energy-dependent, $U(E_k)$, results in rather complicated integral equations for the dual states, which do not have the simple structure of the L-S equation for the energy-independent potential.

For the purpose of completeness we derive below the integral equations for the dual states $|\widetilde{\psi}_{\vec{k}}^{(+)}\rangle$ and $|\widetilde{\psi}_{\vec{k}}^{(-)}\rangle$, when the interaction $U(E_k)$ is used. This is easily accomplished by using Eqs. (5) and (6) in conjunction with the L-S equations for $|\psi_{\vec{k}}^{(+)}\rangle$ and $|\psi_{\vec{k}}^{(-)}\rangle$, Eqs. (3) and (4),

$$\langle \widetilde{\Psi}_{\vec{k}'}^{(+)} | \vec{k}\rangle = (2\pi)^3\, \delta(\vec{k}' - \vec{k}) - \int \frac{d\vec{q}}{(2\pi)^3} \cdot$$

$$\cdot \frac{\langle \widetilde{\Psi}_{\vec{k}'}^{(+)} | \vec{q}\rangle \langle \vec{q} | U(E_k) | \Psi_{\vec{k}}^{(+)}\rangle}{E_k - E_q + i\varepsilon} \qquad (9)$$

$$\langle \vec{k}' | \tilde{\Psi}_{\vec{k}}^{(-)} \rangle = (2\pi)^3 \delta(\vec{k}'-\vec{k})$$

$$- \int \frac{d\vec{q}}{(2\pi)^3} \frac{\langle \Psi_{\vec{k}'}^{(-)} | U(E_{k'}) | \vec{q} \rangle \langle \vec{q} | \tilde{\Psi}_{\vec{k}}^{(-)} \rangle}{E_{k'} - E_q + i\varepsilon} \quad (10)$$

Clearly, there is no simple way to recast the above two equations into an operator L-S equations such as Eq. (3) and (4). To show how in the limit of energy-independent potential one recovers Eqs. (12), (14) of I for $|\tilde{\psi}_k^{(+)}\rangle$ and $|\tilde{\psi}_k^{(-)}\rangle$, we derive another relation. We take for example $|\psi_{\vec{k}}^{(+)}\rangle$. Again from Eq. (5) and using the following equivalent L-S equation for $|\psi_{\vec{k}}^{(+)}\rangle$

$$| \psi_{\vec{k}}^{(+)} \rangle = | \vec{k} \rangle + G^{(+)}(E_k, U(E_k)) \, U(E_k) \, | \vec{k} \rangle \quad (11)$$

where $G^{(+)} = (E_k - T - U(E_k) + i\varepsilon)^{-1}$, we have, after writing $\langle \tilde{\psi}_{\vec{k}'}^{(+)} | \vec{k} \rangle = (2\pi)^3 \delta(\vec{k}'-\vec{k}) + \langle \tilde{\phi}_{\vec{k}'}^{(+)} | \vec{k} \rangle$,

$$\left[\langle \tilde{\phi}_{\vec{k}'}^{(+)} | \vec{k} \rangle + \frac{1}{E_k - E_{k'} + i\varepsilon} \langle \tilde{\psi}_{\vec{k}'}^{(+)} | U(E_k) | \vec{k} \rangle \right]$$

$$= \Delta_{\vec{k}',\vec{k}} \quad (12)$$

where

$$\Delta_{\vec{k}',\vec{k}} \equiv - \langle \tilde{\psi}_{\vec{k}'}^{(+)} | G_o^{(+)}(E_k) \left[T(E_k, U(E_k)) - T(E_k, U(E_{k'})) \right] G_o^{(+)}(E_k) U(E_k) | \vec{k} \rangle \quad (13)$$

In Eq. (13), the T-matrices $T(E_k, U(E_k))$ and $T(E_k, U(E_{k'}))$ are given by

$$T(E_k, U(E_k)) = U(E_k) + U(E_k) G_o^{(+)}(E_k) T(E_k, U(E_k)) \quad (14)$$

$$T(E_k, U(E_{k'})) = U(E_{k'}) + U(E_{k'}) G_o^{(+)}(E_k) T(E_k, U(E_{k'})) \quad (15)$$

The unphysical second T-matrix is introduced here merely to simplify the notation and should be understood only through its Born series.

When $U(E_k)$ is taken to be independent of E_L, $\Delta_{\vec{k}\vec{k}'}$ identically vanishes and we have

$$\langle \tilde{\phi}_{\vec{k}'}^{(+)} | \vec{k} \rangle = - \frac{1}{E_k - E_{k'} + i\varepsilon} \langle \tilde{\psi}_{\vec{k}'}^{(+)} | U | \vec{k} \rangle$$

$$= \frac{1}{E_{k'} - E_k + i\varepsilon} \langle \tilde{\psi}_{\vec{k}'}^{(+)} | U | \vec{k} \rangle$$

$$= \langle \tilde{\psi}_{\vec{k}'}^{(+)} | U \left(G_o^{(+)}(E_{k'}) \right)^\dagger | \vec{k} \rangle$$

$$= \langle \tilde{\Psi}_{\vec{k}'}^{(+)} | [G_o^{(+)}(\varepsilon_{k'}) \, U^\dagger]^\dagger \, | \vec{k} \rangle \qquad (16)$$

And accordingly

$$| \tilde{\Psi}_{\vec{k}'}^{(+)} \rangle = | \vec{k}' \rangle + G_o^{(+)}(\varepsilon_{k'}) \, U^\dagger \, | \Psi_{\vec{k}'}^{(+)} \rangle \qquad (17)$$

In general, $\Delta_{\vec{k},\vec{k}}$ is not zero and the one has to settle with the integral equations (9) and (10). In the following section we avoid the explicit use of $U(E_k)$ and work instead with its equivalent-energy-independent version.

III. THE CASE OF ENERGY-DEPENDENT POTENTIALS

So far in our discussion we have taken the complex potential to be energy-independent. Of course, as we have discussed in I. The microscopic optical potential is intrinsically energy-dependent. It would therefore be important to extend our discussion in the previous sections to this case. In order to exploit fully the formalism developed in I for energy-independent interactions, it is advantageous not to work with the energy-dependent potential directly as we have seen in

Section II but rather with its equivalent energy-independent optical potential (EEIOP).

Several recent papers have addressed the question of how to construct the EEIOP. This potential, by construction, generates the same full scattering wave function as the original energy-dependent interaction. Although in practice the EEIOP presents more problems than advantages[2], we shall use it here in a formal sense.

From the defining relations

$$U(\varepsilon_k) | \Psi_{\vec{k}}^{(+)} \rangle = \bar{U}_+ \, | \Psi_{\vec{k}}^{(+)} \rangle \qquad (18)$$

and

$$\langle \Psi_{\vec{k}}^{(-)} | \, U(\varepsilon_k) = \langle \Psi_{\vec{k}}^{(-)} | \, \bar{U}_- \qquad (19)$$

and the completeness relations of the $\psi^{(+)}$ and $\psi^{(-)}$, we can write down immediately the formal expressions

$$\bar{U}_+ = \int \frac{d\vec{k}}{(2\pi)^3} \, U(\varepsilon_k) \, | \Psi_{\vec{k}}^{(+)} \rangle \langle \Psi_{\vec{k}}^{(+)} | \qquad (20)$$

$$\bar{U}_- = \int \frac{d\vec{k}}{(2\pi)^3} \, | \Psi_{\vec{k}}^{(-)} \rangle \langle \Psi_{\vec{k}}^{(-)} | \, U(\varepsilon_k) \qquad (21)$$

In general the two matrix elements in Eqs (20),(21) are different.

To show this we use the general time invariance transformation properties of these matrix elements. Namely

$$\langle \vec{q} \mid \bar{U}_+ \mid \vec{p} \rangle = \langle \vec{q} \mid T^\dagger T \, \bar{U}_+ \mid \vec{p} \rangle^*$$
$$= \langle \vec{q} \mid T^\dagger T \, \bar{U}_+ T^{-1} T \mid \vec{p} \rangle^*$$

$$= \langle \vec{p} \mid \left(T \, \bar{U}_+ T^{-1} \right)^\dagger \mid \vec{q} \rangle \qquad (22)$$

On the other hand

$$\langle \vec{q} \mid \bar{U}_+ \mid \vec{p} \rangle = \int \frac{d\vec{k}}{(2\pi)^3} \langle \vec{q} \mid T^\dagger T \, U(\epsilon_k) \mid \Psi_{\vec{k}}^{(+)} \rangle^* \cdot$$
$$\cdot \langle \tilde{\Psi}_{\vec{k}}^{(+)} \mid T^\dagger T \, \mathbb{1} \mid \vec{p} \rangle^*$$

$$= \int \frac{d\vec{k}}{(2\pi)^3} \langle \Psi_{\vec{k}}^{(-)} \mid U(\epsilon_k) \mid \vec{q} \rangle \langle \vec{p} \mid \tilde{\Psi}_{\vec{k}}^{(-)} \rangle$$

$$= \int \frac{d\vec{k}}{(2\pi)^3} \langle \vec{p} \mid \tilde{\Psi}_{\vec{k}}^{(-)} \rangle \langle \Psi_{\vec{k}}^{(-)} \mid U(\epsilon_k) \mid \vec{q} \rangle$$

$$= \langle \vec{p} \mid \bar{U}_- \mid \vec{q} \rangle \qquad (23)$$

or

$$\bar{U}_+ = \bar{U}_-^T$$

The above is true in the absence of spin-orbit interaction.

Assuming thus no spin-orbit interaction we can write instead of Eqs. (11)-(14) of I, the following

$$\mid \Psi_{\vec{k}}^{(+)} \rangle = \mid \vec{k} \rangle + G_o^{(+)}(\epsilon_k) \, \bar{U}_+ \mid \Psi_{\vec{k}}^{(+)} \rangle \qquad (25)$$

$$\mid \tilde{\Psi}_{\vec{k}}^{(+)} \rangle = \mid \vec{k} \rangle + G_o^{(+)}(\epsilon_k) \, \bar{U}_+^\dagger \mid \Psi_{\vec{k}}^{(+)} \rangle \qquad (26)$$

$$\mid \Psi_{\vec{k}}^{(-)} \rangle = \mid \vec{k} \rangle + G_o^{(-)}(\epsilon_k) \left(\bar{U}_+ T \right)^\dagger \mid \Psi_{\vec{k}}^{(-)} \rangle \qquad (27)$$

$$\mid \tilde{\Psi}_{\vec{k}}^{(-)} \rangle = \mid \vec{k} \rangle + G_o^{(-)}(\epsilon_k) \, \bar{U}_+^T \mid \tilde{\Psi}_{\vec{k}}^{(-)} \rangle \qquad (28)$$

With Eqs. (25)-(28), we can immediately derive the equivalent to Eqs. (42), (43) with U substituted by $\bar{U}_+$ ($\bar{U}_-$) inside matrix elements involving $\mid \psi^{(+)} \rangle$ ($\mid \psi^{(-)} \rangle$). Using the original definitions of $\bar{U}_+$ and $\bar{U}_-$, we recover Eqs. (42), (43) with $U \to U(E_k)$. It remains to be seen how the inclusion of spin-orbit interaction modifies the above results.

IV. CONCLUSIONS

In this paper we have extended the discussion in I to energy-dependent optical potentials. This has been accomplished through the construction of two equivalent-energy-dependent potentials connected with the ingoing and outgoing wave

boundary conditions. In the absence of spin-orbit interaction the two potentials are found to be the transposed of each other. The unitarity defect relation involving $\left(S^+S - 1\right)$ is then straightforwardly generalized to energy-dependent potentials.

REFERENCES

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