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ON THE OPERATOR SOLUTION OF THE LIOUVILLE THEORY

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Abstract

Using quantized self dual fields, we present an explicit operator solution to the Liouville theory, and discuss the results.

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The Liouville theory[1] is defined by the action

$$S = \int d^2x \{\frac{1}{2}(\partial_\mu \phi)^2 + \mu_0 e^\phi\} \tag{1}$$

with the field equation

$$\partial^2 \phi + \mu_0 e^\phi = 0 \tag{2}$$

This theory presents many problems; it is known to be integrable, and we aim at an explicit solution. In the literature there are excelent works, dealing with operator solutions to this theory [2][3].We claim that the present solution has the advantage to give the Liouville field ϕ directly in terms of known fields, whose Fourier expansion is well defined and is given in terms of creation and annihilation operators. In [2] a quantum bäcklund transformation is constructed, providing a nonlinear relation between the Liouville field and a free field. In [3] a solution for the model in a strip of width π is presented, a problem relevant to open strings. It is very important, especially in string theory[4], to have an explicit solution to the Liouville theory, since it describes the conformal anomaly, if one shifts the dimension away from the critical value[4].

The classical theory has been studied in considerable detail [3][5]. The point we will use in the construction, is the inverse map

$$e^\phi = \frac{u'v'}{(u-v)^2} \tag{3}$$

where u is a right mover and v a left mover

$$(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})u = 0 \tag{4a}$$

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})v = 0 \tag{4b}$$

which represent action angle variables in the hamiltonian formalism, obtained from the change of variables

$$\begin{pmatrix}\phi\\ \pi\end{pmatrix} \to \begin{pmatrix}u\\ v\end{pmatrix} \tag{5a}$$

where π is the conjugate momentum

$$\pi(t,x) = \frac{\partial\phi(t,x)}{\partial t} \tag{5b}$$

The Poisson Bracket

$$\{\phi(t,x), \pi(t,y)\} = \gamma\delta(x-y) \tag{6}$$

leads to well defined Poisson Bracket relations for u and v [3][6]:

$$\{u(x), u(y)\} = \gamma\epsilon(x-y)[u(x)-u(y)]^2 + \gamma[u(x)^2 - u(y)^2] \tag{7a}$$

$$\{v(x), v(y)\} = -\gamma\epsilon(x-y)[v(x)-v(y)]^2 + \gamma[v(x)^2 - v(y)^2] \tag{7b}$$

$$\{u(x), u(y)\} = 2\gamma[u(x)v(y) - v(y)^2] \tag{7c}$$

There is an Ansatz which simplifies the Poisson Bracket structure very much. It is

$$p(t,x) = \frac{d}{dx} ln \frac{1}{\sqrt{u'}} = -\frac{1}{2}\frac{u''}{u'} \tag{8}$$

in terms of which we have

$$\{p(x), p(y)\} = \gamma\delta'(x-y) \tag{9}$$

Moreover, p is a left moving field

$$(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})p(t,x) = 0 \tag{10}$$

This system can be quantized by methods recently employed[7].The hamiltonian can be written in terms of the field

$$s = p^2 + p' \tag{11}$$

as

$$H = \frac{1}{\gamma}\int dx[s[u] + s[v]] \tag{12}$$

Where $s[v]$ is obtained substituting the left moving field u by the right moving v in (14), obtaining a right moving q(t,x), such that

$$q(t,x) = \frac{d}{dx} ln \frac{1}{\sqrt{v'}} = -\frac{1}{2}\frac{v''}{v'} \tag{13}$$

$$\{q(x), q(y)\} = \gamma\delta'(x-y) \tag{14}$$

and

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})q(t,x) = 0 \tag{15}$$

With the hamiltonian (12) we have the relation

$$\dot{s} = \{H, s\} = s'. \tag{16}$$

Therefore, in terms of the fields p(t,x) and q(t,x) we have

$$H = \frac{1}{\gamma}\int dx\{p(x)^2 + q(x)^2\} \tag{17}$$

The action is given by

$$S = \frac{1}{2}\int dx \int dy p(x)\epsilon(x-y)p(y) - \int dx p(x)^2 + \frac{1}{2}\int dx \int dy q(x)\epsilon(x-y)q(y) + \int dx q(x)^2 \tag{18}$$

The solution to the above model is given in terms of elementary operators[7]. The lagrangean describes constrained fields[8]. We have the following decompositions at t=0

$$p(x) = -i\int_0^\infty dk\sqrt{\frac{k}{\pi}}\left[a(k)e^{-ikx} - a^\dagger(k)e^{ikx}\right] \tag{19}$$

$$q(x) = -i\int_0^\infty dk\sqrt{\frac{k}{\pi}}\left[b(k)e^{-ikx} - b^\dagger(k)e^{ikx}\right] \tag{20}$$

where

$$\left[a(k), a^\dagger(k')\right] = \delta(k-k') \tag{21}$$

$$\left[b(k), b^\dagger(k')\right] = \delta(k-k') \tag{22}$$

A conformal algebra is generated by $L(x) = \frac{1}{2}p(x)^2$ as

$$\left[L(x), L(y)\right] = i\left[L(x) + L(y)\right]\delta'(x-y) - \frac{i}{24\pi}\delta'''(x-y) \tag{23}$$

Another algebra is generated for $\tilde{L}(x) = q(x)^2$. We are able to solve the problem for u and v in terms of p and q:

$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty} dy\epsilon(x-y)exp\left(-\int_{-\infty}^{\infty}\epsilon(y-z)p(z)dz\right) \tag{24}$$

$$v(x) = \frac{1}{2}\int_{-\infty}^{\infty} dy\epsilon(x-y)exp\left(-\int_{-\infty}^{\infty}\epsilon(y-z)q(z)dz\right) \tag{25}$$

At this point one should notice the existence of two arbitrary constants appearing in the solution of (8) (resp. (13)) and in the integration of u' (resp. v'). At the classical level they are harmeless due to the Möbius invariance, which in terms of u reads

$$u \to u_M = \frac{a_{11}u + a_{12}}{a_{21}u + a_{22}} \tag{26}$$

with an analogous expression for v. Thus, we may fix arbitrariness as above (eqs. 24,25). However this is not the end of the story. In the case of the slution discussed in [3], there is a q-number constant multiplying the exponential, which is fixed requiring due commutation relations for u. We shall not dwell on this issue much longer, refering to a forthcoming publication on the type of commutation relations obeyed by u and v, but we keep in mind an arbitrariness at this point in our present solution.

The full solution to the Liouville theory is thus given by

$$e^\phi = \frac{exp\left\{-\int_{-\infty}^{\infty}\epsilon(x-y)\left[p(t,y)+q(t,y)\right]dy\right\}}{\left\{\frac{1}{2}\int_{-\infty}^{\infty}dy\epsilon(x-y)\left(e^{-\int_{-\infty}^{\infty}\epsilon(y-z)p(t,z)dz} - e^{-\int_{-\infty}^{\infty}\epsilon(y-z)q(t,z)dz}\right)\right\}^2} \tag{27}$$

or also

$$\phi(t,x) = -\int_{-\infty}^{\infty}\epsilon(x-y)\left[p(t,y)+q(t,y)\right]dy$$

$$-2ln\left\{\frac{1}{2}\int_{-\infty}^{\infty}dy\epsilon(x-y)\left(e^{-\int_{-\infty}^{\infty}\epsilon(y-z)p(t,z)dz} - e^{-\int_{-\infty}^{\infty}\epsilon(y-z)q(t,z)dz}\right)\right\} \tag{28}$$

The integrals are easily computed

$$\chi(x) = \frac{1}{2}\int_{-\infty}^{\infty}\epsilon(x-y)p(t,y)dy =$$

$$-\int_0^\infty \frac{dk}{\sqrt{k\pi}}\left[a(k)e^{ikx} + a^\dagger(k)e^{-ikx}\right] \tag{29}$$

The field $\chi(x)$ is described by a local lagrangean

$$\mathcal{L} = \frac{1}{2}\chi'\dot{\chi} - \frac{1}{2}\chi'^2 \tag{30}$$

However, canonical quantization[7][8] leads to a non local commutator

$$[\chi(x), \chi(y)] = -\frac{i}{2}\epsilon(x-y) \tag{31}$$

The infrared properties of the field χ present problems, since its decomposition is similar to a bosonic massless field in two dimensions [9], eq.(29). The vacuum of the theory, when defined in terms of the creation/annihilation operators $a^{\dagger}(k)$ and $a(k)$ as

$$a(k)|0>=0 \tag{32}$$

presents therefore severe infrared problems. The field $\phi(t,x)$is very complicated, and the computation of the vacuum expectation value of it depends on the infrared cut-off., since the commutator

$$[\chi^{-}(x), \chi^{+}(y)] = \int_0^{\infty} \frac{dk}{k\pi} e^{ik(x-y)} \tag{33}$$

is divergent.

We hope to have answers to these problems in the future.

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