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BOSONIZATION METHODS IN STRING THEORY

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Abstract

I discuss the use of bosonization/fermionization techniques to convert non-linear operators of one theory into linear operators of the dual. We generalize non abelian bosonization to the case where the central charge of the Kac-Moody algebra is not unity. In particular, using this generalization of non-abelian bosonization, the bosonic string vertex of the compactified theory, turns out to be the fundamental field of the fermionic theory, or bound states of it thus permitting explicit computations easily.

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1. General equivalence between WZW theory and Thirring model.

Two dimensional models and conformal invariance are very important ingredients in the modern description of strings[1]. In two dimensional space time the conformal group in infinite dimensional, being equivalent to the group of reparametrizations, supporting its relevance to string theory[2].

Bosonization methods have been used frequently in string theories. In particular, the equivalence between the Neveu-Schwarz-Ramond and the Green-Schwarz formulations of superstring theories can be understood most appropriately applying the above methods to obtain relations between the three different representations of the little group of the Lorentz group, which in ten dimensions, being SO(8) has a three-fold automorphism relating them[3].

Our aim in this communication is to use this same technique in a different way, though yet in the context of string theory but this time in the purely bosonic case. We wish to fermionize the purely bosonic string theory defined in a group manifold, a problem related to the compactification issue in string theory. Following Gepner and Witten[4], we define the latter through the Wess-Zumino-Witten (WZW) action[5]. We argue that this action is equivalent to the non abelian Thirring model[6], with the corresponding symmetry group, at the non trivial fix point[7][8]. The action of a sigma model with a Wess Zumino term is invariant under a given group G. It describes compactification of the string, which turns out to have a non abelian symmetry as well, rather than $U(1)^d$, due to solitons wrapping around the tori[9]. It is well known, on the other hand, that the non abelian determinant can be written in terms of the WZW theory[10]. This is the root of non abelian bosonization. When the level of the representation is k=1, the WZW theory is equivalent to free fermions. This has been shown by different methods [5][11][12]. We first consider this issue in a more general scheme, namely general values of k, and a more elaborated interacting fermionic theory.

The bosonic theory (WZW) is defined by the action

$$S = \frac{k}{16\pi}\int d^2x\partial^\mu g^{-1}\partial^\mu g + \frac{k}{8\pi}\int dt\int d^2x\epsilon_{\mu\nu}g^{-1}\dot{g}g^{-1}\partial^\mu g g^{-1}\partial^\nu g \quad (1.1)$$

The conserved currents are

$$J_{+ij}(x) = N[i(g^{-1}\partial_+ g)_{ij}(x)] \quad (1.2a)$$

$$J_{-ij}(x) = N[i(\partial_- g g^{-1})_{ij}(x)] \quad (1.2b)$$

obeying

$$\partial_- J_+ = 0 = \partial_+ J_- \quad (1.3)$$

In the usual bosonization prescription, where k=1, the group valued field g_{ij} is given in terms of fermions as

$$g_{ij}(x) = \mu^{-1}N[\psi^\dagger_{1i}(x)\psi_{2j}(x)] \quad (1.4)$$

In order to study the case $k \neq 1$ we need further information. We consider the U(n) invariant Thirring model, which is known to have a conformally invariant solution at a certain value of the coupling constant[6], as we will see.The model is defined by the lagrange density

$$\mathcal{L} = \overline{\psi}i\partial\psi - \frac{g}{2}\overline{\psi}\gamma^\mu\tau^a\psi\overline{\psi}\gamma^\mu\tau^a\psi - \frac{g'}{2}\overline{\psi}\gamma^\mu\psi\overline{\psi}\gamma^\mu\psi \quad (1.5)$$

where $[\tau^a,\tau^b] = if^{abc}\tau^c$, and the field equation is

$$i\partial\psi = gj^a\tau^a\psi + g'j\psi \quad (1.6a)$$

We define the currents

$$j^a_\mu = \overline{\psi}\gamma^\mu\tau^a\psi \quad (1.6b)$$

$$j_\mu = \overline{\psi}\gamma^\mu\psi \quad (1.6c)$$

They are Noether currents of SU(n) and charge conservation, respectively, while $j^5_\mu = \epsilon_{\mu\nu}j^\nu$ is associated to pseudocharge conservation. However the current $j^{a5}_\mu = \epsilon_{\mu\nu}j^{\nu a}$ fails to be conserved if the field equations are used

$$\partial_\mu j^{\mu a5} = gf^{abc}j^{\lambda b}j^{c5}_\lambda \quad (1.7)$$

Nonetheless, Dashen and Frishman[6] were able to show that there is a particular value of the coupling constant were vacuum fluctuations change the above equation. We study the conditions under which the quantum model exhibits conformal invariance. First consider the equal time commutators

$$[j_0^a(t,x), j_0^b(t,y)] = if^{abc} j_0^c(t,x)\delta(x-y) \quad (1.8a)$$

$$[j_0^a(t,x), j_1^b(t,y)] = if^{abc} j_1^c(t,x)\delta(x-y) + i\frac{k}{2\pi}\delta^{ab}\delta'(x-y) \quad (1.8b)$$

$$[j_1^a(t,x), j_1^b(t,y)] = if^{abc} j_0^c(t,x)\delta(x-y) \quad (1.8c)$$

If the theory is scale invariant, it follows from the above equation that j_μ^a has scale dimension one. We can prove now that j_μ^a is divergenceless and curl free. To see this consider the two point function

$$< 0|j_+(x_+,x_-)j_+(y_+,y_-)|0 >= \frac{C}{(x_+ - y_+ + i\epsilon)^2} \quad (1.9)$$

The right hand side is fixed by the fact that $j_+ = j_0 + j_1$ transforms under Lorentz as $1/x_+$. We may analogously consider functions of j_-. From those expressions it follows that

$$\partial_\mu j^\mu = 0 \quad (1.10a)$$

as well as

$$\epsilon^{\mu\nu}\partial_\mu j_\nu = 0 \quad (1.10b)$$

The following commutators can be computed now

$$[j_\pm^a(x_\pm), j_\pm^b(y_\pm)] = if^{abc} j_\pm^c(x_\pm)\delta(x_\pm - y_\pm) + i\frac{k}{2\pi}\delta^{ab}\delta'(x_\pm - y_\pm) \quad (1.11a)$$

$$[j_\pm(x_\pm), j_\pm(y_\pm)] = iC_0\delta'(x_\pm - y_\pm) \quad (1.11b)$$

One finds also

$$[j_\pm, \psi(y)] = -(a \pm \overline{a}\gamma^5)\psi(y)\delta(x_\pm - y_\pm) \quad (1.11c)$$

$$[j_\pm^a, \psi(y)] = -\sigma\frac{1}{2}\lambda^a(1 \pm \delta\gamma^5)\psi(y)\delta(x_\pm - y_\pm) \quad (1.11d)$$

where, due to Jacobi identity $\sigma = 1$ and $\delta^2 = 1$. It is very important to notice that the energy momentum tensor is of the Sugawara form

$$T(x_\pm) = \frac{1}{2C_0} : j_\pm(x_\pm)^2 : + \frac{\pi}{n+k} : j_\pm^a(x_\pm)^2 : \quad (1.12)$$

This implies that the Fourier coefficients of the energy momentum tensor satisfy a Virasoro Kac-Moody algebra, together with the Fourier coefficients of the currents.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m} \quad (1.13a)$$

$$[L_n, J_m^a] = -mJ_{n+m}^a \quad (1.13b)$$

$$[J_m^a, J_n^b] = if^{abc}J_{n+m}^c + nk\delta^{ab}\delta_{n,-m} \quad (1.13c)$$

where the central charge of the Virasoro algebra is determined, and is given by [13]

$$c = \frac{k dim G}{c_V + k} \qquad (1.14)$$

. For the SU(n) case it reads $c_{SU(n)} = \frac{k(n^2-1)}{n+k}$.

The (conformal invariant) two point function is given in terms of the spin-s and anomalous dimension- γ as is generally the case in conformally invariant theories

$$< 0|\psi_1(x)\psi_1{}^\dagger(y)|0 >=$$

$$f_0\{i(x_+ - y_+) + \epsilon\}^{-2s}\{-(x-y)^2 + i\epsilon(x_+ + x_- - y_+ - y_-)\}^{-(d-s)} \qquad (1.15)$$

The four point function has several constraints, arising from the fermionic field equations. These have been set up by Dashen and Frishman, by means of the following construction: the energy momentum tensor is of the Sugawara form, (1.12), hence using (1.11) one finds the commutator of the energy momentum tensor with the fermionic field. Since this tensor generates translations, we identify the commutator with the derivative of ψ, obtaining:

$$i\partial^-\psi_1 = \frac{1}{2}\{2\pi\frac{1-\delta}{n+k}\frac{1}{2}\lambda^b : j_-^b\psi_1 : +\frac{a-\bar{a}}{C_0} : j_-\psi_1 :\} \qquad (1.16a)$$

$$i\partial^+\psi_2 = \frac{1}{2}\{2\pi\frac{1-\delta}{n+k}\frac{1}{2}\lambda^b : j_+^b\psi_2 : +\frac{a-\bar{a}}{C_0} : j_+\psi_2 :\} \qquad (1.16b)$$

$$i\partial^+\psi_1 = \frac{1}{2}\}2\pi\frac{1+\delta}{n+k}\frac{1}{2}\lambda^b : j_+^b\psi_1 : +\frac{a+\bar{a}}{C_0} : j_+\psi_1 :\} \qquad (1.16c)$$

$$i\partial^-\psi_2 = \frac{1}{2}\{2\pi\frac{1+\delta}{n+k}\frac{1}{2}\lambda^b : j_-^b\psi_2 : +\frac{a+\bar{a}}{C_0} : j_-\psi_2 :\} \qquad (1.16d)$$

At this point we identify $g = \frac{2\pi(1-\delta)}{n+k}$, $\delta = 1$ corresponds to the free theory, and $\delta = -1$ a non trivial fix point. The four point function is written in terms of a scaling piece times a function of the Möbius invariant variables u and v given by

$$u = \frac{[i(x_+ - x'_+) + \epsilon][i(y_+ - y'_+) + \epsilon]}{[i(x_+ - y'_+) + \epsilon][i(y_+ - x'_+) + \epsilon]} \qquad (1.17a)$$

$$v = \frac{[i(x_- - x'_-) + \epsilon][i(y_- - y'_-) + \epsilon]}{[i(x_- - y'_-) + \epsilon][i(y_- - x'_-) + \epsilon]} \qquad (1.17b)$$

We write this Green function as follows

$$< 0|\psi_1^a(x)\psi_1^{a'\dagger}(y)\psi_1^b(x')\psi_1^{b'\dagger}(y')|0 >= f_0^2 u^{\gamma+s}v^{\gamma-s}$$

$$\{[i(x_+-y_+)+\epsilon][i(x'_+-y'_+)+\epsilon]\}^{-(\gamma+s)}\{[i(x_--y_-)+\epsilon][i(x'_--y'_-)+\epsilon]\}^{-(\gamma-s)}$$

$$G_{aa'bb'}(u,v) \qquad (1.18)$$

The function G may be written in terms of invariants of the symmetry group as

$$G_{aa'bb'}(u,v) = \delta_{aa'}\delta_{bb'}\tilde{H}_1(u,v) + \delta_{ab'}\delta_{ba'}\tilde{H}_2(u,v) \qquad (1.19)$$

Differential equations are obeyed by H_{12} as a consequence of (1.16).At this point it is convenient to rewrite the four point function, for the sake of comparison with a bosonic theory. We write it in terms of the euclidian variable $z = x_1 - ix_2$

$$< 0|\psi_1^a(z_1)\psi_1^{b\dagger}(z_2)\psi_1^{c\dagger}(z_3)\psi_1^d(z_4)|0 >=$$
$$\left[(z_1 - z_4)(z_2 - z_3)\right]^{-2\Delta}\{\delta^{ab}\delta^{cd}A_1(x) + \delta^{ac}\delta^{bd}A_2(x)\} \quad (1.20)$$

where x is the Möbius invariant combination $x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$.

The functions A_1 and A_2 obey hypergeometric differential equations, due to the fact that the fermionic fields obey the equations of motion, which in terms of complex variables is given by

$$\frac{\partial}{\partial z}\psi_1(z) = \frac{1}{n+k} : j^a\tau^a\psi_1(z) : \quad (1.21)$$

The above implies differential equations for the euclidian four point function. The last bit of information comes from the short distance expansion

$$j^a(z)\psi_1(\omega) = \frac{\tau^a}{z-\omega}\psi_1(\omega) \quad (1.22)$$

which may be used inside the correlators, following BPZ[14] and KZ[15] to define the normal product in (1.21). Thus we obtain

$$\frac{\partial}{\partial z} < 0|\psi_1^a(z_1)\psi_1^{b\dagger}(z_2)\psi_1^{c\dagger}(z_3)\psi_1^d(z_4)|0 >= \ < 0|\psi_1^a(z_1)\psi_1^{b\dagger}(z_2)\psi_1^{c\dagger}(z_3)\psi_1^d(z_4)|0 >$$
$$\frac{2}{n+k}\{\frac{\tau^a_{ii'}\tau^a_{j'j}\delta_{kk'}\delta_{ll'}}{z_2 - z_1} + \frac{\tau^a_{ii'}\tau^a_{k'k}\delta_{jj'}\delta_{ll'}}{z_3 - z_1} + \frac{\tau^a_{ii'}\tau^a_{ll'}\delta_{kk'}\delta_{jj'}}{z_1 - z_4}\} \quad (1.23)$$

From the above one arrives at differential equations which should be fulfilled by the A's:

$$x(x-1)\frac{\partial A_1}{\partial x} = \{(x-1)\frac{\frac{1}{n} - n}{2(n+k)} + \frac{1}{2n(n+k)}x\}A_1 - (x-1)\frac{1}{2(n+k)}A_2 \quad (1.24a)$$

$$x(x-1)\frac{\partial A_2}{\partial x} = \{x\frac{\frac{1}{n} - n}{2(n+k)} + \frac{1}{2n(n+k)}(x-1)\}A_2 - x\frac{1}{2(n+k)}A_1 \quad (1.24b)$$

The solutions of this equations are obtained in terms of hypergeometric functions as (crossing symmetry is required to determine h as below)

$$A_1(x) = \mathcal{F}_1^{(0)}(x) + h\mathcal{F}_1^{(1)}(x) \quad (1.25a)$$

$$A_2(x) = \mathcal{F}_2^{(0)}(x) + h\mathcal{F}_2^{(1)}(x) \quad (1.25b)$$

where the functions F are given by

$$\mathcal{F}_1^{(0)}(x) = x^{-2\Delta}(1-x)^{\Delta_1 - 2\Delta}F(-\frac{1}{n+k}, \frac{1}{n+k}, 1 + \frac{1}{n+k}; x) \quad (1.26a)$$

$$\mathcal{F}_2^{(0)}(x) = -(2n+k)^{-1}x^{1-2\Delta}(1-x)^{\Delta_1 - 2\Delta}F(1 - \frac{1}{n+k}, 1 + \frac{1}{n+k}, 2 + \frac{1}{n+k}; x) \quad (1.26b)$$

$$\mathcal{F}_1^{(1)}(x) = x^{\Delta_1 - 2\Delta}(1-x)^{\Delta_1 - 2\Delta}F(-\frac{n-1}{n+k}, -\frac{n+1}{n+k}, 1 - \frac{1}{n+k}; x) \quad (1.26c)$$

$$\mathcal{F}_2^{(0)}(x) = -nx^{\Delta_1 - 2\Delta}(1-x)^{\Delta_1 - 2\Delta}F(-\frac{n-1}{n+k}, -\frac{n+1}{n+k}, -\frac{n}{n+k}; x) \quad (1.26d)$$

and the function F(a,b,c;x) is the hypergeometric function defined by the series expansion

$$F(a, b, c; x) = 1 + \frac{ab}{1!c}x + \frac{a(a+1)b(b+1)}{2!c(c+1)}x^2 + \ldots \quad (1.27a)$$

and the constant h is

$$h = \frac{1}{n^2} \frac{\Gamma\left(\frac{n-1}{n+k}\right)\Gamma\left(\frac{n+1}{n+k}\right)\left\{\Gamma\left(\frac{k}{n+k}\right)\right\}^2}{\Gamma\left(\frac{k+1}{n+k}\right)\Gamma\left(\frac{k-1}{n+k}\right)\left\{\Gamma\left(\frac{n}{n+k}\right)\right\}^2} \tag{1.27b}$$

We compare these results with those obtained by Knizhnik and Zamolodchikov[15] for the correlators of the WZW theory. Actually, both theories have solutions derived from analogous equations. Let us first of all present the KZ solution of the WZW theory, which is by now well known. We have the two point function given by

$$< g_{ij}(z,\overline{z}) g^{-1}_{kl}(\omega,\overline{\omega}) >= \delta_{ik}\delta_{jl}(z-\omega)^{-2\Delta}(\overline{z}-\overline{\omega})^{-2\Delta} \tag{1.28}$$

which, after comparison with (1.15), corresponds to the identification

$$< g_{ij}(z,\overline{z}) g^{-1}_{kl}(\omega,\overline{\omega}) >=< \psi_1^{i\dagger}(z)\psi_1^k(\omega) >< \psi_2^{l\dagger}(\overline{\omega})\psi_2^l(\overline{z}) > \tag{1.29}$$

or, in terms of operators, it corresponds to (1.4). However, in order to interpret (1.4) as an operator identity between WZW field g and Dashen Frishman fermions, we proceed to the four point function. This has been computed and reads

$$\begin{aligned}
&< g_{ii'}(z_1,\overline{z}_1) g^{-1}_{jj'}(z_2,\overline{z}_2) g^{-1}_{kk'}(z_3,\overline{z}_3) g_{ll'}(z_4,\overline{z}_4) >= \\
&\mu^{-1}(z_{12}z_{23}\overline{z}_{14}\overline{z}_{23})^{-2\Delta}\Big\{\left[\mathcal{F}_1^{(0)}(x)\mathcal{F}_1^{(0)}(\overline{x}) + h\mathcal{F}_1^{(1)}(x)\mathcal{F}_1^{(1)}(\overline{x})\right]\delta_{ij}\delta_{kl}\delta_{i'j'}\delta_{k'l'} + \\
&\qquad\left[\mathcal{F}_1^{(0)}(x)\mathcal{F}_2^{(0)}(\overline{x}) + h\mathcal{F}_1^{(1)}(x)\mathcal{F}_2^{(1)}(\overline{x})\right]\delta_{ij}\delta_{kl}\delta_{i'k'}\delta_{j'l'} + \\
&\left[\mathcal{F}_2^{(0)}(x)\mathcal{F}_1^{(0)}(\overline{x}) + h\mathcal{F}_2^{(1)}(x)\mathcal{F}_1^{(1)}(\overline{x})\right]\delta_{ik}\delta_{jl}\delta_{i'j'}\delta_{k'l'} + \\
&\qquad\left[\mathcal{F}_2^{(0)}(x)\mathcal{F}_2^{(0)}(\overline{x}) + \mathcal{F}_2^{(1)}(x)\mathcal{F}_2^{(1)}(\overline{x})\right]\delta_{ik}\delta_{jl}\delta_{i'k'}\delta_{j'l'}\Big\}
\end{aligned} \tag{1.30}$$

Now, comparison of the above with fermionic correlators confirm the result (1.4) as interpreted above, after using crossing symmetry.

2. Conformal theories and strings

After identification of both bosonic and fermionic theories, we discuss their common framework, namely conformal invariance.

Both theories have conserved quantum currents. The WZW theory was constructed in such a way that there is a right moving and a left moving current, which in euclidian domain means an antiholomorphic and a holomorphic currents:

$$\overline{J}^a(\overline{z}) = tr g^{-1}\partial_{\overline{z}} g \tau^a \tag{2.1a}$$

$$J^a(z) = tr \partial_z g g^{-1}\tau^a \tag{2.1b}$$

In the case of SU(n) Thirring model, this property is not valid already at classical level because of the interaction, since $\partial_\mu j^{\mu a 5} = g f^{abc} j^{b\mu} j^{c5}_\mu$. However, as we discussed in the quantum case the rhs vanishes due to conformal invariance at the fix point. In this case we have also holomorphic and antiholomorphic free currents $J^a(z)$ and $\overline{J}^a(\overline{z})$. The equations of motion in this language are

$$\frac{1}{2}(n+k)\frac{\partial}{\partial z} g(z,\overline{z}) =: J^a(z)\tau^a g(z,\overline{z}) : \tag{2.2a}$$

$$\frac{1}{2}(n+k)\frac{\partial}{\partial \bar{z}}g(z,\bar{z}) =: \bar{J}^a(\bar{z})\tau^a g(z,\bar{z}) : \tag{2.2b}$$

where, using the operator product expansion (OPE) we have

$$J^a(\omega)\tau^a g(z,\bar{z}) = \frac{c_g}{\omega - z}g(z,\bar{z}) + \sum_{n=1}^{\infty}(\omega - z)^{n-1}\tau^a J^a_{-n} g(z,\bar{z}) \tag{2.3}$$

where $\tau^a\tau^a = c_g 1$.This allows us to define

$$: J^a(z)\tau^a g(z,\bar{z}) := \lim_{\omega\to z}\big(J^a(\omega) - \frac{\tau^a}{\omega - z}\big)g(z,\bar{z}) \tag{2.4}$$

Actually, this allows for the determination of critical exponents, since the equation of motion may be regarded as associating the field

$$\chi = \big(J^a_{-1}\tau^a - \frac{1}{2}(n+k)L_{-1}\big)g \tag{2.5}$$

to a null field, in analogy with the construction of the Verma modulus in the Virasoro algebra[14][15].

For the fermionic theory, on the other hand we have, at the non trivial fix point

$$\partial_z\psi_2(z,\bar{z}) = 0 \tag{2.6a}$$

$$\frac{1}{2}(n+k)\partial_z\psi_1(z,\bar{z}) =: J^a(z)\tau^a\psi_1(z) : \tag{2.6b}$$

$$\partial_{\bar{z}}\psi_1(z,\bar{z}) = 0 \tag{2.6c}$$

$$\frac{1}{2}\partial_{\bar{z}}\psi_2(z,\bar{z}) =: \bar{J}^a(\bar{z})\tau^a\psi_2(z,\bar{z}) : \tag{2.6b}$$

which are again the equivalent of (2.4) , and ψ_1 and ψ_2 are a fermionic representation of the constraint (2.5).

At the string theory level, describing the bosons in terms of fermions avoids difficulties associated to the description of the compactification process[16][7]. This procedure implies an action containing an antissymmetric tensor coupled to the bosons given by

$$S = \int d^2x\{\frac{1}{2}g^{\mu\nu}\partial_a X_\mu \partial_b X_\nu g^{ab} + \frac{1}{2}B^{\mu\nu}\partial_a X_\mu \partial_b X_\nu \epsilon^{ab}\} \tag{2.7}$$

which is rather involved. But the above WZW type action is equivalent to a fermionic model as we discussed. Moreover, vertex operators in the bosonic language

$$V_\alpha = e^{i\alpha X} \tag{2.8}$$

are given by a product of fermionic basic field operators, due to the bosonization/fermionization rule

$$\psi = e^{iX} \tag{2.9}$$

Thus correlation functions of vertices may be computed as the product of the well known vertices in Minkowski space, times Thirring field correlators, the latter describing the compactified piece of the theory.

In order to study properties of fermionization of strings we start with the field constraints imposed by compactification

First we study the abelian case where the compactified manifold is a torus, with $[U(1)]^d$ symmetry. Thus, the separation between left and right movers proceeds as usual, and the equivalent Thirring model is abelian, described by the action

$$\frac{1}{2\pi}\int d^2\xi\left[i\overline{\psi}\gamma^\mu\partial_\mu\psi - H_{ij}\psi_1^{\dagger i}\psi_1^i\psi_2^{\dagger j}\psi_2^j\right] \tag{2.10}$$

with field equations

$$\frac{\partial}{\partial x_+}\psi_1^i = -2\pi i F_{ai}J_+^a(x_+)\psi_1 \tag{2.11a}$$

$$\frac{\partial}{\partial x_-}\psi_2^i = -2\pi i K_{ai}J_-^a(x_-)\psi_2 \tag{2.11b}$$

where $H_{ij} = F_{ai}K_{aj}$.

Notice that there is no relation among the various U(1) coupling constants.

The right and left movers and euclidian holomorphic and antiholomorphic position operators are given by

$$X^a(z) = X_0^a + \frac{i}{2}p_0 lnz + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_{-n}^a z^{-n} \tag{2.12a}$$

$$\overline{X}^a(z) = \overline{X}_0^a - \frac{i}{2}p_0 ln\overline{z} + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_{-n}^a \overline{z}^{-n} \tag{2.12b}$$

The position operator is obtained adding the two expressions above,

$$X(z,\overline{z}) = X(z) + \overline{X}(\overline{z}) \tag{2.13}$$

We also define

$$\tilde{X}(z,\overline{z}) = X(z) - \overline{X}(\overline{z}) \tag{2.14}$$

In a compactified space, we have symmetries associated to the period of non trivial closed orbits; in the present case we shall, for the time being restrain to compactification on a d dimensional torus, with a common radius R.Thus we have for each $X^a(z,\overline{z})$ a symmetry

$$X \to X + 2\pi R \tag{2.15}$$

This symmetry is rather intuitive; but it is not the only one. Consider the mode expansion on a torus

$$X = X_0 + \frac{m}{R}\tau + 2LR\sigma + \frac{i}{2}\sum_{n\neq 0}(\alpha_n z^{-n} + \tilde{\alpha}_n \overline{z}^{-n})\frac{1}{n} \tag{2.16}$$

The momentum p_0 is quantized in unities of $\frac{1}{R}$, corresponding to the above symmetry. On the other hand the momentum conjugate to X, p_0, has eigenvalue 2LR, thus a multiple or 2R. This means that X_0 lies on a circle of radius $1/2R$; therefore the symmetry

$$X \to X + \frac{\pi}{R} \tag{2.17}$$

is obeyed.

The fermionic field constructed out of X and $\tilde{X}$ is given , in the case of a single U(1) by

$$\psi_{\alpha,\beta}(z,z) = e^{i\alpha X_L(z)+i\beta X_R(z)} = e^{i\frac{\alpha+\beta}{2}X \quad +i\frac{\alpha-\beta}{2}\tilde{X}} \tag{2.18}$$

and under (2.15),(2.17) transforms as

$$\psi_{\alpha,\beta} \to \psi_{\alpha,\beta}e^{i(\alpha+\beta)\pi R} \tag{2.19a}$$

$$\psi_{\alpha,\beta} \to \psi_{\alpha,\beta}e^{i(\alpha+\beta)\frac{\pi}{2R}} \tag{2.19b}$$

One should note at this point that (2.19a) and (2.19b) correspond, in terms of strings and moduli space, to modular transformations. Thus, modular invariant amplitudes requires that physically relevant operators be invariant under the above transformations. Those are either bilinears of the type $\psi\dagger(x)\psi(x)$, or bound states $\left(\psi_{\alpha,\beta}(x)\right)^F := \psi_{F\alpha,F\beta}(x)$, such that

$$F(\alpha+\beta)R = 2n \tag{2.20a}$$

$$F(\alpha-\beta)\frac{1}{2R} = 2m \tag{2.20b}$$

where F,n,m, are integers.This implies, for the spin, the relation

$$s = \frac{\lambda}{2} = \frac{\alpha^2-\beta^2}{8} = \frac{mn}{F^2} \tag{2.21}$$

which is a rational number.

We can work out in some detail the free field case, $\beta = 0$, $\lambda = \frac{\alpha^2}{4}$. For spin one, $\lambda = 2$, $\psi_{\alpha,0}$ is a physical field alone for $R = \sqrt{2}$, since ψ is invariant under (2.19a,b) in these circunstances. For half integer spins, we need R=1 and bound states of two fermions.

In general we have more complicated relations. The energy momentum tensor is of the Sugawara form

$$\theta_+(x_+) = \frac{\pi}{n+k} : \left\{J^a(x_+)\right\}^2 : \tag{2.22}$$

in the U(n) case. The commutation relations between currents and elementary fields are

$$[J^a_-(x), \psi^i_1(y)] = -\frac{1}{2}A^{ai}\psi^i_1\delta(x_- - y_-) \tag{2.23a}$$

$$[J^a_+(x), \psi^i_1(y)] = -\frac{1}{2}B^{ai}\psi^i_1\delta(x_+ - y_+) \tag{2.23b}$$

$$[J^a_+(x), \psi^i_2(y)] = -\frac{1}{2}C^{ai}\psi^i_2\delta(x_+ - y_+) \tag{2.23c}$$

$$[J^a_-(x), \psi^i_2(y)] = -\frac{1}{2}D^{ai}\psi^i_2\delta(x_- - y_-) \tag{2.23d}$$

Thus, using the Sugawara form of the energy momentum tensor we find

$$\begin{aligned}
\left[\theta_-(x_-), \psi^i_1(x'_-, x'_+)\right] &= -2\pi A^{ai} : J^a_-(x_-)\psi^i_1 : \delta(x_- - x'_-) + \frac{i}{4}A^{ai}A^{ai}\psi^i_1\delta'(x_- - x'_-) \\
\left[\theta_+(x_+), \psi^i_1(x'_-, x'_+)\right] &= -2\pi B^{ai} : J^a_+(x_+)\psi^i_1 : \delta(x_+ - x'_+) + \frac{i}{4}B^{ai}B^{ai}\psi^i_1\delta'(x_+ - x'_+) \\
\left[\theta_+(x_+), \psi^i_2(x'_-, x'_+)\right] &= -2\pi C^{ai} : J^a_+(x_+)\psi^i_2 : \delta(x_+ - x'_+) + \frac{i}{4}C^{ai}C^{ai}\psi^i_2\delta'(x_+ - x'_+) \\
\left[\theta_-(x_-), \psi^i_2(x'_-, x'_+)\right] &= -2\pi D^{ai} : J^a_-(x_-)\psi^i_2 : \delta(x_- - x'_-) + \frac{i}{4}D^{ai}D^{ai}\psi^i_2\delta'(x_- - x'_-)
\end{aligned} \tag{2.24}$$

For the ψ 's to satisfy the equations of motion we need $B^{ai} = 2F^{ai}$ and $D^{ai} = 2K^{ai}$. Demanding also that ψ has spin $s = \frac{\lambda}{2}$ we have

$$\sum_a A^{ai}A^{ai} - \sum_i B^{ai}B^{ai} =$$

$$\sum_a C^{ai}C^{ai} - \sum D^{ai}D^{ai} = 4\lambda \quad (2.25)$$

There is a possible solution to these equations given by:

$$A^{ai} = \left(Z^{ai} + \tilde{Z}^{ai} - Y^{ab}Z^{bi}\right)\sqrt{\lambda} \qquad (2.26a)$$

$$B^{ai} = \left(Z^{ai} - \tilde{Z}^{ai} + Y^{ab}Z^{bi}\right)\sqrt{\lambda} \qquad (2.26b)$$

$$C^{ai} = \left(Z^{ai} + \tilde{Z}^{ai} + Y^{ab}Z^{bi}\right)\sqrt{\lambda} \qquad (2.26c)$$

$$D^{ai} = \left(Z^{ai} - \tilde{Z}^{ai} - Y^{ab}Z^{bi}\right)\sqrt{\lambda} \qquad (2.26d)$$

where Z is symmetric, $Z = Z^{-1}$ and $Y^{ab} = -Y^{ba}$.

The fermionic operators are given by the usual Mandelstam formulae

$$\psi_1^i = e^{-iA^{ai}X_-^a(x_-)+iB^{ai}X_+^a(x_+)} (2.27a)$$

$$\psi_2^i = e^{iC^{ai}X_+^a(x_+)-iD^{ai}X_-^a(x_-)} (2.27b)$$

If we consider now that the torus is obtained dividing the space by a lattice Λ generated by the vectors E_μ^a, we have the symmetry

$$X^a \rightarrow X^a + 2\pi E_\mu^a n^\mu \qquad (2.28)$$

where n^μ are integers, while the dual lattice $\tilde{E}_\mu^a$, defined by the relation

$$\tilde{E}_\mu^a E^{\nu a} = \delta_\mu^\nu \qquad (2.29)$$

generate the symmetry

$$\tilde{X}^a \rightarrow \tilde{X}^a + \pi \tilde{E}_\mu^a m^\mu \qquad (2.30)$$

For the fermionic fields these transformations act as

$$\psi_{12}^i \rightarrow \psi_{12}^i e^{\mp 2\pi i(\tilde{Z}^{ai} \mp Y^{ab}Z^{bi} \pm B^{ab}Z^{bi})E_\mu^a n^\mu} \qquad (2.31a)$$

and

$$\psi_{12}^i \rightarrow \psi_{12}^i e^{-\pi i Z^{ai}\tilde{E}_\mu^a m^\mu} \qquad (2.31b)$$

These twists are rather complicated; modular invariant operators must be defined in an invariant way. Some particular cases have been analised in (16) but the general discussion is too complicated.

Therefore, in order to obtain modular invariant operators, we need to consider bound states of the above defined operators. The most important ones we consider in this work are the vertex operators. The first is the tachyon vertex

$$V(x) = e^{ikX(x)} \qquad (2.32)$$

which in a compactified theory can be written as a product of the Minkowski space piece

$$V_{Mink}(x) = e^{ik_\mu X^\mu(x)} \qquad (2.33)$$

where $\mu = 0, \ldots, d_{Mink} - 1$ are unbounded coordinates,times the compactified piece

$$V_{Comp}(x) = e^{ik_i X^i(x)} \tag{2.34}$$

$i = d_{Mink}, \ldots, D - 1$ are compactified coordinates. According to Gepner and Witten, we have to consider for the compactified part of the vertex

$$V_{Comp} = g_{ij}(z.\bar{z}) \tag{2.35}$$

which is equivalent to the above expression after using non abelian fermionisation, and abelian bosonisation prescriptions.

With these preliminaries out of the way we pass to the discussion of the non abelian case which is far more important, since non abelian symmetry groups appear naturally in compactification processes, such as that defined in heterotic strings compactification, namely 26 right moving bosonic coordinates turn into 10 open and 16 compactified, the latter having symmetry group $E(8) \times E(8)$.

The complete operator solution of the non abelian Thirring model is not known.Nevertheless, there are helpful expressions which may be used in some bound state Green functions.For the product of spinor and antispinor field operators we have

$$\psi_1^i(x)\psi_1^j(y) = C(x_+ - y_+)^{-A}(x_- - y_-)^{-B}$$
$$e^{-\frac{i}{2C_0}\left\{(a+\bar{a})\int_{x_+}^{y_+} j_+(w_+)dw_+ + (a-\bar{a})\int_{x_-}^{y_-} j_-(w_-)dw_-\right\}} M^{ij}(x,y) \quad (2.36)$$

where M satisfies

$$\partial^+ M = 0 \tag{2.37a}$$

Again, we have the non abelian fermionization and abelian bosonization formulae. Comparing abelian and non abelian cases, defined on the same compactification torus, we have the identifications

$$H_{ij} \longleftrightarrow (\tau^a)_{ii}(\tau^a)_{jj} \tag{2.38a}$$

or

$$F_{ai} \longleftrightarrow (\tau^a)_{ii}$$
$$K_{ai} \longleftrightarrow (\tau^a)_{jj} \tag{2.38b}$$

Also

$$A^{ai} \longleftrightarrow (\tau^a)^{ii} \tag{2.38c}$$

$$C^{ai} \longleftrightarrow (\tau^a)^{ii} \tag{2.38d}$$

It follows that that for an even self dual lattice ψ is modular invariant, and there are no further constraints in the non abelian piece. Only abelian pieces leave arbitrariness.

We discuss now the definition of those bound states which are relevant to the computation of vertices. As it turns out, a vertex is a product of the compactified piece times the Minkowski space part,eqs. (2.33) to (2.35).The compactified piece consists of products of elementary fermionic fields. We shall consider a bound state

$$j^{ab}(\xi) = N[\psi^a(\xi)\psi^{b\dagger}(\xi)] \tag{2.39}$$

We have an explicit formula for the four point function

$$< \psi^a(\xi + \epsilon)\psi^{b\dagger}(\xi)\psi^{c\dagger}(\xi')\psi^d(\xi' + \epsilon') > \tag{2.40}$$

given by(1.23).We compute (2.40) for $\epsilon\epsilon' \to 0$, using

$$F(-\frac{1}{n+k}, \frac{1}{n+k}, 1+\frac{n}{n+k}; x) \simeq 1 - \frac{x}{(n+k)(2n+k)} \tag{2.41}$$

In the above limit we have

$$< \psi^a(\xi+\epsilon)\psi^{b\dagger}(\xi)\psi^{c\dagger}(\xi')\psi^d(\xi'+\epsilon') >= \delta^{ab}\delta^{cd}(\epsilon\epsilon')^{-2\Delta}$$
$$+h\{\frac{\epsilon\epsilon'}{(\xi-\xi')^2}\}^{\Delta_1-2\Delta}(\delta^{ab}\delta^{cd} - n\delta^{ac}\delta^{bd})$$
$$-\frac{(\epsilon\epsilon')^{1-2\Delta}}{(\xi-\xi')^2}(\frac{k-n}{nk(n+k)}\delta^{ab}\delta^{cd} - k\delta^{ac}\delta^{bd}) + \dots \tag{2.42}$$

The first contribution is trivial, and must be subtracted. For $k \neq 1$, the second contribution is the only one remaining after renormalization is performed. We have in this case

$$< N[\psi^a(\xi)\psi^{b\dagger}(\xi)][\psi^{c\dagger}(\xi')\psi^d(\xi')] >= \frac{h\mu^{-8\Delta}}{(\xi-\xi')^{\frac{2}{n(n+k)}}}(\delta^{ab}\delta^{cd} - n\delta^{ac}\delta^{bd}) \tag{2.43}$$

Therefore we have an anomalous dimension $\gamma_j = \frac{1}{n(n+k)}$.

For k=1, on the other hand, we have $h = 0$, and j has canonical dimension (1) since

$$< j^{ab}(\xi)j^{dc}(\xi') >= \frac{\mu^{-8\Delta}}{(\xi-\xi')^2}(\frac{k-n}{nk(n+k)}\delta^{ab}\delta^{cd} - k\delta^{ac}\delta^{bd}) \tag{2.44}$$

and the problem gives nontrivial results for $k \neq 1$.

3.Conclusions.

We analised the boson/fermion equivalence for non abelian conformal invariant theories with central charge not equal to one. As it turns out, the fermionic model has a quantum conformal invariant Thirring coupling. Correlators may be computed in a closed form and are representations of the conformal algebra.

For string theories, this construction is relevant for computations involving vertex operators. Moreover, for closed strings on a compact manifold there are in general constraints on the compactified piece of the vertex operator.For a non abelian symmetry, as we have seen these constraints may not interfere if the lattice is adequate, but if abelian symmetries are involved one must form adequate bound states in order to ensure modular invariance. Correlators involving vertices may be computed without much difficulty.

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