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THIRRING STRINGS: USE OF GENERALIZED NON ABELIAN BOSONIZATION TECHNIQUES

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ABSTRACT

A discussion of compactified bosonic string theory is presented, with a thorough use of conformal invariance in order to relate the theory to the WZW model and U(n) invariant Thirring model at critical coupling. The quantization of these theories is discussed, as well as the definition of vertex operators in the various equivalent models above.

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1. INTRODUCTION: CONFORMAL INVARIANT STRING THEORIES IN A COMPACTIFIED SPACE

In the last few years strings have been proved to be extremely important objects for the description of fundamental interactions(1). However there are many technical difficulties in the description of string dynamics, and maybe one of the most relevant aspects to be fully understood is the compactification issue. In this note it is my aim to relate the string defined on a compactified space time to a fermionic model, in such a way that in the latter, complicated and at the same time important operators such as the vertices(1)(2), turn out to be elementary fields. Therefore, correlators involving vertices, which are non linear in the string field variables, will turn out to be linear, in terms of fermionic variables.

The action describing a string on a compactified manifold is given by

$$S = \frac{1}{2\pi}\int d\sigma\, d\tau\, \partial_\alpha X^a \partial_\beta X^b \left(\eta^{\alpha\beta} g_{ab}(x) + \varepsilon^{\alpha\beta} B_{ab}(x)\right) \tag{1.1}$$

where the first term is the usual Polyakov-string action, whereas the latter is the Wess-Zumino term, necessary to mantain conformal invariance in the compactified space(3)(4); it is necessary since the first action describes a non linear sigma model, which is, a in general asymptotically free(5).

The separation between left and right movers is rather subtle in this theory. In order to substantiate this statement we elaborate further. In the conformal theory, left and right movers are the equivalent to holomorphic and antiholomorphic fields in the euclidian version of the theory[6].

We have

$$X^a_\pm = X^a_{0\pm} + P^a_{0\pm}\, x_\pm + \sum_{n\neq 0} \alpha^a_{n\pm}\, e^{in x_\pm} \tag{1.2}$$

where the position variable is simply the sum of left and right movers

$$X^a(x) = X^a_+(x_+) + X^a_-(x_-) \tag{1.3}$$

One defines also the dual field $\tilde{X}^a$, given by the expression

$$\tilde{X}^a(x) = X^a_+(x_+) - X^a_-(x_-) + B^{ab}(x)\left(X^a_+(x_+) + X^a_-(x_-)\right) \tag{1.4}$$

motivated by the algebra valued fields of WZW theory

$$J^+(x_+) = g^{-1}\partial^+ g \tag{1.5}$$

$$J^-(x_-) = \partial^- g\, g^{-1} \tag{1.6}$$

where X^a is the Lie algebra valued field corresponding to the group valued g- field described by the action

$$S = \frac{1}{8\pi}\int d^2x\, \mathrm{tr}\, \partial_\mu \tilde{g}^{-1}\partial^\mu \tilde{g} + \frac{1}{4\pi}\int_0^1 dr \int d^2x\, \mathrm{tr}\, \varepsilon^{\mu\nu}\, \tilde{g}^{-1}\partial_\mu \tilde{g}\, \tilde{g}^{-1}\partial_\nu \tilde{g}\, \tilde{g}^{-1}\partial_r \tilde{g} \tag{1.7}$$

We can formally identify

$$J^{\pm a}(x_\pm) = \mp P^{a\pm}(x_\pm) = -\frac{1}{\pi}\, \partial^\pm X^a_\pm \tag{1.8}$$

If the symmetry group is abelian, as for example $[U(1)]^d$, a torus, we may suppose all X^a's to be independent. Their commutation relations are easily derived from J's Kac-Moody algebra, and are given by[7]

$$[X^a_+(x_+), X^b_+(y_+)] = -\frac{i}{4}\pi\, \delta^{ab}\, \varepsilon(x_+ - y_+) \tag{1.9a}$$

$$[P^a_+(x_+), X^b_+(y_+)] = -\frac{i}{2}\delta^{ab}\,\delta(x_+ - y_+) \tag{1.9b}$$

$$[J^a_+(x_+), J^b_+(y_+)] = \frac{i}{2\pi}\,\delta^{ab}\,\delta'(x_+ - y_+) \tag{1.9c}$$

which is an abelian Kac-Moody algebra. We will elaborate further on the abelian theory later on. Now the (non abelian) WZW theory will be discussed in some more detail.

Starting from (1.7), it is possible to formally integrate over the r variable, obtaining an effective action, with an unknown expression $A(g)$[8]:

$$S = \frac{1}{8\pi}\int d^2x \,\mathrm{tr}\, \partial_\mu g^{-1} \partial^\mu g + \frac{1}{4\pi}\int d^2x\, \mathrm{tr}\, A(g)\,\partial_0 g \tag{1.10}$$

For the purpose of canonical quantization, A(g) is not needed. All necessary information is provided by the derivative

$$F_{ij;kl}(g) = \frac{\partial A_{ij}}{\partial g_{lk}} - \frac{\partial A_{kl}}{\partial g_{ji}} \tag{1.11}$$

which is given by

$$F_{ij;kl} = \partial_1 g^{-1}_{il}\, g^{-1}_{kj} - g^{-1}_{il}\, \partial_1 g^{-1}_{kj} \tag{1.12}$$

The momentum conjugate to g is given by

$$\pi_{ij} = \frac{1}{4\pi}\partial_0 g^{-1}_{ji} + \frac{1}{4\pi} A_{ji}(g) \tag{1.13}$$

Define also

$$\hat{\pi}_{ij} = \pi_{ij} - \frac{1}{4\pi} A_{ji} \tag{1.14}$$

Canonical commutation relations have been discussed in the literature; they are[8]:

$$[\pi_{ij}(x), \pi_{kl}(y)] = 0 \tag{1.15a}$$

$$[g_{ij}(x), \pi_{kl}(y)] = i\,\delta_{ik}\,\delta_{jl}\,\delta(x^1-y^1) \tag{1.15b}$$

$$[g_{ij}(x), g_{kl}(y)] = 0 \tag{1.15c}$$

at equal time; it follows that the current, which can be written in terms of the elementary field as

$$J_+^a = \frac{-i}{4\pi}\mathrm{tr}\, g^{-1}\partial_+ g\, \tau^a = \mathrm{tr}\left(i\,\hat{\pi} g - \frac{i}{4\pi} g^{-1}\partial_1 g\right)\tau^a \tag{1.16}$$

(there is also an expression for J_-^a) has well defined commutation relations. Actually since the above expression is purely left moving the expression below is valid for any time:

$$[J_+^a(x), J_+^b(y)] = i f^{abc} J_+^c(x)\,\delta(x-y) + \frac{i}{2\pi}\delta^{ab}\delta'(x-y) \tag{1.17}$$

The energy momentum tensor may be readily computed, being of the Sugawara form[4]

$$\Theta_+(x_+) = \frac{\pi}{c_v + k} :\left(J^a(x_+)\right)^2: \tag{1.18}$$

We define the field operator

$$X^a(x) = \int_{-\infty}^{x} J^a(y)\, dy \tag{1.19}$$

which in view of (1.17) obeys the algebra

$$[X^a(x), X^b(y)] = \frac{i}{2} f^{abc} \left(X^c(x) - X^c(y) \right) \varepsilon(x-y)$$

$$+ \frac{i}{4\pi} \delta^{ab} \varepsilon(x-y) \tag{1.20}$$

Our problem is to implement compactification, namely realize the identifications

$$X^a \equiv X^a + 2\pi E_\mu{}^a n^\mu \tag{1.21}$$

where n^μ are integers, and E_μ^a generate a lattice Λ, in such a way that $\mathbb{R}^d/\Lambda$ is the target manifold of our sigma model described by the action (1.1). A string theory defined on a compact manifold has a further symmetry[7] associated to the dual field $\tilde{X}$ defined in (1.4). This symmetry is related to the dual lattice $\tilde{\Lambda}$, generated by $\tilde{E}_\mu^a$, defined as

$$E_\mu^a \tilde{E}^{a\nu} = \delta_\mu{}^\nu \tag{1.22}$$

in the one dimensional case $E = R$, and the symmetry is

$$X \equiv X + 2\pi R \tag{1.23}$$

In order to understand the second symmetry, consider the mode expansion of the closed string field on a compact space

$$X(x) = X_0 + \frac{M}{R}\tau + 2LR\sigma + \text{oscilators} \tag{1.24}$$

noticing that both zero modes are quantized, the former $\left(P_\tau = \frac{M}{R}\right)$ because the string is in a compact space of radius $1/R$, and P_τ must be quantized (M is integer), and the other one ($P_\sigma = L\,2\,R$, L integer) because σ is defined up to multiples of π, in which case X can only modify by multiples of 2π, eq. (1.23).

Therefore, left and right movers are

$$X_L = X_{0L} + \frac{1}{2}\left(\frac{M}{R} + 2LR\right)(\tau+\sigma) + \text{oscilators} \tag{1.25}$$

$$X_R = X_{0R} + \frac{1}{2}\left(\frac{M}{R} - 2LR\right)(\tau-\sigma) + \text{oscilators} \tag{1.26}$$

implying the expression

$$\tilde{X} = \tilde{X}_0 + \frac{M}{R}\sigma + 2LR\tau + \text{oscilators} \tag{1.27}$$

which has a momentum $P_\tau = L\,2\,R$, quantized in unities of $2R$; thus the identification

$$\tilde{X} \equiv \tilde{X} + \pi/R \tag{1.28}$$

is valid.

2. BOSONIZATION AND FERMIONIZATION IN CONFORMALLY INVARIANT 2-D FIELD THEORIES

The principal non linear σ model with a Wess-Zumino term written as a functional of a group valued field $g(x)$ is described by the action

$$S = \frac{1}{4\lambda^2}\, tr \int d^2x\, \partial_\mu g^{-1} \partial^\mu g + \frac{k}{8\pi}\, \varepsilon^{\mu\nu}\, tr \int dr \int d^2x\, \tilde{g}^{-1} \dot{\tilde{g}}\, \tilde{g}^{-1} \partial_\mu \tilde{g}\, \tilde{g}^{-1} \partial_\nu \tilde{g} \tag{2.1}$$

This is conformally invariant only if the coupling constant is given by[4]

$$\lambda^2 = \frac{4\pi}{k} \tag{2.2}$$

The constant k is quantized (integer) since the topological term is defined up to redefinitions of the extension $g(x) \to \tilde{g}(r,x)$. Different choices of boundaries differ by multiples of 2π. This system has been related to free fermions by several authors, when $k=1$. This so called non abelian bosonization prescription is realized by the identification

$$g_{ij}(x) = \mu^{-1} N \left[\psi^{\dagger}_{1i}(x)\, \psi_{2j}(x) \right] \tag{2.3}$$

where N is a normal product prescription and μ an arbitrary

mass parameter. The resulting theory is a multiplet of free fermions[3][4][9].

Aiming at general values of the central charge k we study the G-invariant Thirring model (we will specialize $G=SU(n)$, when writting explicit formulas). The lagrangean is given by[10]

$$\mathcal{L} = i\bar{\Psi} \not{\partial} \psi - \frac{1}{2} g' \bar{\Psi}\gamma^\mu\psi\, \bar{\Psi}\gamma_\mu\psi - \frac{1}{2} g\, \bar{\Psi}\gamma^\mu\tau^a\psi\, \bar{\Psi}\gamma_\mu\tau^a\psi \tag{2.4}$$

with $\left[\tau^a, \tau^b\right] = i f^{abc} \tau^c$

The formal field equation is

$$i\not{\partial}\psi = g\, j^a_\mu \gamma^\mu \tau^a \psi + g' j_\mu \gamma^\mu \psi \tag{2.5}$$

where the currents are formally defined by the expressions

$$j^a_\mu = \bar{\Psi}\gamma_\mu \tau^a \psi \tag{2.6a}$$

$$j_\mu = \bar{\Psi}\gamma_\mu \psi \tag{2.6b}$$

According to the symmetries of the model, we have three fundamental conservation laws

$$\partial^\mu j_\mu = 0 \tag{2.7a}$$

$$\partial^\mu j^a_\mu = 0 \tag{2.7b}$$

$$\varepsilon^{\mu\nu} \partial_\mu j_\nu = 0 \tag{2.7c}$$

The curl of j^a_μ is however non zero, obeying a "non conservation" law

$$\varepsilon^{\mu\nu} \partial_\mu j^a_\nu = g\, \varepsilon_{\mu\nu} f^{abc} j^{\mu b} j^{\nu c} \tag{2.8}$$

Dashen and Frishman(10) studied the conditions under which the quantum model displays conformal invariance. They considered the equal time commutators

$$\left[j^a_0(t,x), j^b_0(t,y) \right] = i f^{abc} j^c_0(t,x)\, \delta(x-y) \tag{2.9a}$$

$$\left[j^a_0(t,x), j^b_1(t,y) \right] = i f^{abc} j^c_1(t,x)\, \delta(x-y) + \frac{ik}{2\pi} \delta^{ab} \delta'(x-y) \tag{2.9b}$$

$$\left[j^a_1(t,x), j^b_1(t,y) \right] = i f^{abc} j^c_0(t,x) \delta(x-y) \tag{2.9c}$$

If the theory is scale invariant, it follows from above

that j^a_μ has scale dimension one. We can prove now that j^a_μ is divergenceless and curl free. Consider the vector $J_\mu(x)$ with dimension one, and the two point function

$$\langle 0 | J_+(x_+, x_-)\, J_+(y_+, y_-) | 0 \rangle = \frac{\mathbb{C}}{(x_+ - y_+ + i\varepsilon)^2} \tag{2.10a}$$

The right hand side is fixed by the fact that $J_+ = J_0 + J_1$, transforms under Lorentz as $1/x_+$. We may consider analogously

$$\langle 0 | J_-(x_+, x_-)\, J_-(y_+, y_-) | 0 \rangle = \frac{\mathbb{C}}{(x_- - y_- + i\varepsilon)^2} \tag{2.10b}$$

From these expressions it follows that

$$\partial^\mu J_\mu(x) = 0 \tag{2.11a}$$

as well as

$$\varepsilon^{\mu\nu} \partial_\mu J_\nu(x) = 0 \tag{2.11b}$$

Therefore the non conservation law (2.8) transforms, due to quantum fluctuations into a conservation law in the quantum theory. We will verify which conditions are left by the above imposition of conformal invariance.

Let us set up the commutation relations:

i) singlet currents obey an (abelian) Kac-Moody algebra

$$[j_\pm(x_\pm), j_\pm(y_\pm)] = 2i\, \mathbb{C}_0\, \delta'(x-y) \qquad (2.12a)$$

$$[j_+(x_+), j_-(y_-)] = 0 \qquad (2.12b)$$

ii) singlet currents act on fermions in the same way as the abelian Thirring model

$$[j_\pm(x_\pm), \psi(y)] = -(a \pm \bar{a}\gamma_5)\psi(y)\,\delta(x_\pm - y_\pm) \qquad (2.13)$$

iii) Non abelian currents satisfy a Kac-Moody algebra

$$[j_+^a(x_+), j_+^b(y_+)] = 2i f^{abc} j_+^c(x_+)\,\delta(x_+ - y_+) + \frac{ik}{\pi}\delta^{ab}\delta'(x_+ - y_+) \qquad (2.14a)$$

$$[j_-^a(x_-), j_-^b(y_-)] = 2i f^{abc} j_-^c(x_-)\,\delta(x_- - y_-) + \frac{ik}{\pi}\delta^{ab}\delta'(x_- - y_-) \qquad (2.14b)$$

$$[j_+^a(x_+), j_-^b(y_-)] = 0 \qquad (2.14c)$$

iv) Non singlet currents act on fermions as

$$[j_\pm^a(x_\pm), \psi(y)] = -\sigma(1 \pm \delta\gamma_5)\frac{1}{2}\lambda^a\psi(y)\,\delta(x_\pm - y_\pm) \qquad (2.15)$$

where Jacobi identity requires

$$\sigma = 1 \qquad (2.15a)$$

$$\delta^2 = 1 \qquad (2.15b)$$

The energy momentum tensor is of the Sugawara form; fixing the constants requiring that currents and energy momentum tensor satisfy the usual form of Virasoro Kac-Moody algebra we have

$$\Theta_\pm(x_\pm) = \frac{1}{2\,\mathbb{C}_0} :(j_\pm(x_\pm))^2: + \frac{\pi}{C_v + k} :(j_\pm^a(x_\pm))^2: \qquad (2.17)$$

The constant $\mathbb{C}_0$ defined in (2.12a) is arbitrary, and will depend, as we shall see, on the dimension and spin of the fermionic field. The Casimir C_v is given by the relation

$$f^{abc} f^{dbc} = C_v\,\delta^{ad} \qquad (2.18)$$

For SU(n) we have

$$C_v = n \qquad (2.19)$$

and k is the central charge of the Kac-Moody algebra; thus it is an integer.

The energy momentum tensor satisfies the Virasoro algebra:

$$\left[\theta_\pm(x_\pm), \theta_\pm(y_\pm)\right] = 2i\left(\theta_\pm(x_\pm) + \theta_\pm(y_\pm)\right)\delta'(x_\pm - y_\pm)$$
$$-\frac{i}{6\pi}\delta'''(x_\pm - y_\pm) \tag{2.20}$$

where the central charge is[11]

$$c = \frac{k \dim G}{c_v + k} = \frac{k(n^2-1)}{n+k} \tag{2.21}$$

the last expression has been especialized for $G = SU(n)$.

Using (2.13) and (2.15), the action of the energy momentum tensor (2.17) on the fermionic field may be computed. On the other hand, we know that it generates translations. Therefore, equations of motion may be computed. They are

$$i\partial^-\psi_1 = \frac{1}{2}\left\{2\pi\frac{1-\delta}{c_v+k}\,\frac{1}{2}\lambda^b : j_-^b\,\psi_1 : + \frac{a-\bar{a}}{c_0} : j_-\psi_1 :\right\} \tag{2.22a}$$

$$i\partial^+\psi_2 = \frac{1}{2}\left\{2\pi\frac{1-\delta}{c_v+k}\,\frac{1}{2}\lambda^b : j_+^b\,\psi_2 : + \frac{a-\bar{a}}{c_0} : j_+\psi_2 :\right\} \tag{2.23}$$

$$i\partial^+\psi_1 = \frac{1}{2}\left\{2\pi\frac{1+\delta}{c_v+k}\,\frac{1}{2}\lambda^b : j_+^b\,\psi_1 : + \frac{a+\bar{a}}{c_0} : j_+\psi_1 :\right\} \tag{2.24}$$

$$i\partial^-\psi_2 = \frac{1}{2}\left\{2\pi\frac{1+\delta}{c_v+k}\,\frac{1}{2}\lambda^b : j_-^b\,\psi_2 : + \frac{a+\bar{a}}{c_0} : j_-\psi_2 :\right\} \tag{2.25}$$

Comparing to the formal field equations, we make the identifications

$$g' = \frac{a-\bar{a}}{c_0} \tag{2.26}$$

$$g = 2\pi\frac{1-\delta}{c_v+k} \tag{2.27}$$

Notice that $\delta = 1$ corresponds to the abelian Thirring model ($g = 0$). The point $\delta = -1$, or

$$g = \frac{4\pi}{c_v+k} \tag{2.28}$$

corresponds to a non trivial zero of the β-function, which has not been seen in other treatments (there have been recently some hints in this direction, by the path integral procedure[12]).

Notice also that there is a doubling of the field equations. This is necessary, because the formal expressions (2.6) can no longer be used in view of (2.11), which substituted the formal equation (2.8). Thus (2.24) and (2.25) are interpreted as definitions of j_+^a by ref. (10). Moreover, if $j_\pm^a$, $j_\pm$ are free fields, as predicted by conformal invariance (see (2.7), (2.8)) the equations obeyed by ψ_1 and ψ_2 decouple and the system may be solved. Finally, at the non trivial coupling $(\delta=-1)$ we may adjust the constants $a, \bar{a}$ such that (2.24), (2,25) as holomorphic (antiholomorphic) conditions to be obeyed by ψ_1 and ψ_2 in the euclidianized version.

From this point we specialize to the case of $G = SU(n)$, and compute the two and four point functions. Conformal invariance is enough to compute two point functions

$$\langle 0|\psi_1(x)\psi_1^+(y)|0\rangle = \left[i(x_+-y_+)+\varepsilon\right]^{-2s}\left[-(x-y)^2+i\varepsilon(x_0-y_0)\right]^{s-\gamma} \tag{2.29}$$

where γ is the dimension and s the spin.

The spin and dimension may be computed in terms of the previously defined parameters a, $\bar{a}$, C_0, C_v and k:

$$s = \frac{1}{2}\left[\frac{a\bar{a}}{\pi C_0} - \frac{n^2-1}{n(k+n)}\right] \tag{2.30}$$

$$\gamma = \frac{1}{4\pi}\left[\frac{a^2+\bar{a}^2}{C_0} + 2\pi\frac{n^2-1}{n(n+k)}\right] \tag{2.31}$$

In order to compute functions of ψ_2, interchange x_+, y_+ into x_-, y_-.

In order to compute the four point function we need use of the field equations (2.22-25). The last piece of information comes from the normal product of the current with the elementary field ψ. One uses the commutation relations

$$\left[j_+^{a(-)}(x_+), \psi_1(y)\right] = \frac{1+\delta}{2\pi}\frac{1}{2}\lambda^a\psi(y)\frac{1}{i(x_+-y_+)-\varepsilon} \tag{2.32}$$

and others arising from (2.13), (2.15) and the separation from creation and annihilation operators

$$j_\pm^a(x_\pm) = \int_0^\infty dp\left(a_\pm^a(p)\,e^{-ipx_\pm} + a_\pm^{a+}(p)\,e^{ipx_\pm}\right) \tag{2.33}$$

Therefore, taking the derivative of the correlator

$$\mathcal{F}_{ii',jj'}(x,x',y,y') = \left\langle \psi_1^i(x)\,\psi_1^{i'}(x')\,\psi_1^{+j}(y)\,\psi_1^{+j'}(y')\right\rangle \tag{2.34}$$

with respect to x_+ one gets correlators involving the product $\frac{1}{2}\lambda^a\psi_1^i(x)\,j_+^{a(-)}(x)$; at this point one uses the commutator of $j^{a(-)}$ with the remaining fields, to obtain a differential equation for the correlator $\mathcal{F}$. Since this is technically straightforward but long computation, we refere to (13)(14) for details, and write down the results. In order to compare to known results of conformal theory, we write down the results for the euclidian theory. In terms of the z, $\bar{z}$ variables which correspond to $e^{\tau\pm i\varphi}$, the field equations read

$$i\partial_z\psi_1(z,\bar{z}) = \frac{\pi}{2}\frac{1-\delta}{c_v+k}\lambda^b : J_-^b(z)\,\psi_1(z,\bar{z}) + \frac{1}{2}\frac{a-\bar{a}}{C_0} : J(z)\,\psi_1(z,\bar{z}) : \tag{2.35}$$

$$i\partial_{\bar{z}}\psi_1(z,\bar{z}) = \frac{\pi}{2}\frac{1+\delta}{c_v+k}\lambda^b : \bar{J}^b(\bar{z})\,\psi_1(z,\bar{z}) : + \frac{1}{2}\frac{a+\bar{a}}{C_0} : \bar{J}(\bar{z})\,\psi_1(z,\bar{z}) : \tag{2.36}$$

and analogous formulae for ψ_2. The above equations mean that ψ_1 (analogously ψ_2) is a representation of the simplest Verma module[15] of the non abelian theory[16] in terms of fermions since the constraint is

$$\left(J^a_{-1}\,\tau^a + \frac{1}{2\pi}(n+k)\,L_{-1}\right)\psi = 0 \tag{2.37}$$

The ansatz for the four point function is now[10] [13]

$$\langle \psi_1^a(z_1)\,\psi_1^{b+}(z_2)\,\psi_1^{c+}(z_3)\,\psi_1^d(z_4)\rangle =$$

$$= \left[(z_1-z_4)(z_2-z_3)\right]^{-2\Delta}\left\{\delta^{ab}\delta^{cd}A_1(x) + \delta^{ac}\delta^{bd}A_2(x)\right\} \tag{2.38}$$

where

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_4)} \tag{2.38a}$$

is invariant under modular transformations (generated by $L_0, L_{\pm 1}$). The functions $A_{1,2}(x)$ obey, as a result of the fermionic field equations together with (2.37) (specialized to $G = SU(n)$):

$$x(x-1)\frac{dA_1}{dx} = \left[(x-1)\frac{\frac{1}{n}-n}{2(n+k)} + \frac{x}{2n(n+k)}\right]A_1 - (x-1)\frac{1}{2(n+k)}A_2 \tag{2.39a}$$

$$x(x-1)\frac{dA_2}{dx} = \left[x\,\frac{\frac{1}{n}-n}{2(n+k)} + \frac{x-1}{2n(n+k)}\right]A_2 - \frac{x}{2(n+k)}A_1 \tag{2.39b}$$

The solution is given in terms of hypergeometric functions

$$A_1(x) = \mathcal{F}_1^{(0)}(x) + h\,\mathcal{F}_1^{(1)}(x) \tag{2.40a}$$

$$A_2(x) = \mathcal{F}_2^{(0)}(x) + h\,\mathcal{F}_2^{(1)}(x) \tag{2.40b}$$

where

$$\mathcal{F}_1^{(0)}(x) = x^{-2\Delta}(1-x)^{\Delta_1-2\Delta}\,F\left(-\frac{1}{\lambda},\ \frac{1}{\lambda},\ 1+\frac{n}{\lambda};\ x\right) \tag{2.41a}$$

$$\mathcal{F}_2^{(0)}(x) = -\frac{x^{1-2\Delta}}{\lambda+n}(1-x)^{\Delta_1-2\Delta}\,F\left(1-\frac{1}{\lambda},\ 1+\frac{1}{\lambda},\ 2+\frac{n}{\lambda};\ x\right) \tag{2.41b}$$

$$\mathcal{F}_1^{(1)}(x) = x^{\Delta_1-2\Delta}(1-x)^{\Delta_1-2\Delta}\,F\left(-\frac{n-1}{\lambda},\ -\frac{n+1}{\lambda},\ 1-\frac{n}{\lambda};\ x\right) \tag{2.41c}$$

$$\mathcal{F}_2^{(1)}(x) = -n\, x^{\Delta_1 - 2\Delta} (1-x)^{\Delta_1 - 2\Delta} F\left(-\frac{n-1}{\lambda}, -\frac{n+1}{\lambda}, -\frac{n}{\lambda} ; x\right) \tag{2.41d}$$

where

$$\lambda = c_v + k = n + k \tag{2.42a}$$

$$\Delta = \frac{n^2-1}{2n(n+k)} \tag{2.42b}$$

$$\Delta_1 = \frac{n}{n+k} \tag{2.42c}$$

and F is the hypergeometric function

$$F(a,b,c;x) = 1 + \frac{ab}{1!\,c} x + \frac{a(a+1)b(b+1)}{2!\; c(c+1)} x^2 + \dots \tag{2.42d}$$

At last, crossing symmetry (interchanging x by 1-x) fixes h to be

$$h = \frac{1}{n^2} \frac{\Gamma\left(\frac{n-1}{n+k}\right) \Gamma\left(\frac{n+1}{n+k}\right) \left\{\Gamma\left(\frac{k}{n+k}\right)\right\}^2}{\Gamma\left(\frac{k+1}{n+k}\right) \Gamma\left(\frac{k-1}{n+k}\right) \left\{\Gamma\left(\frac{n}{n+k}\right)\right\}^2} \tag{2.42e}$$

These results are worth comparing to correlators of the WZW's g -field, obtained, by Knizhnik and Zamolodchikov[16], using $g(x)$ as a primary field obeying

$$\left(J_{-1}^a t^a + \frac{1}{2\pi}(n+k)\, L_{-1}\right) g = 0 \tag{2.43}$$

with a current $J(z)$ (and also $\bar{J}(\bar{z})$)

The results are as follows.

For the two point function

$$\langle g_{ij}(z,\bar z)\, g^{-1}_{kl}(w,\bar w)\rangle = \delta_{ik}\,\delta_{jl}\,(z-w)^{-2\Delta}(\bar z - \bar w)^{-2\Delta} \tag{2.44}$$

a result following only from conformal invariance, and which, comparing to (2.38-42) implies

$$\langle g_{ij}(z,\bar z)\, g^{-1}_{kl}(w,\bar w)\rangle = \langle \psi^+_{1i}(z)\,\psi_{1k}(w)\rangle \langle \psi_{2j}(\bar z)\,\psi^+_{2l}(\bar w)\rangle \tag{2.45}$$

A sufficient condition for (2.45) to be valid is

$$g_{ij}(z,\bar z) = \psi^+_{1i}(z)\,\psi_{2j}(\bar z) \tag{2.46}$$

We check now this relation in the case of four point functions. It is enough to borrow K.Z's formulas

$$\langle g_{ii'}(z_1,\bar z_1)\, g^{-1}_{jj'}(z_2,\bar z_2)\, g^{-1}_{kk'}(z_3,\bar z_3)\, g_{\ell\ell'}(z_4\bar z_4)\rangle =$$

$$\mu^{-8\Delta}(z_{14}z_{23}\bar z_{14}\bar z_{23})^{-2\Delta}\Big\{\Big[\mathcal{F}_1^{(0)}(x)\,\mathcal{F}_1^{(0)}(\bar x)+h\,\mathcal{F}_1^{(1)}(x)\,\mathcal{F}_1^{(1)}(\bar x)\Big]\delta_{ij}\delta_{k\ell}\delta_{i'j'}\delta_{k'\ell'}+$$

$$+\Big[\mathcal{F}_1^{(0)}(x)\,\mathcal{F}_2^{(0)}(\bar x)+h\,\mathcal{F}_1^{(1)}(x)\,\mathcal{F}_2^{(1)}(\bar x)\Big]\delta_{ij}\delta_{k\ell}\delta_{i'k'}\delta_{j'\ell'}+$$

$$+\Big[\mathcal{F}_2^{(0)}(x)\,\mathcal{F}_1^{(0)}(\bar x)+h\,\mathcal{F}_2^{(1)}(x)\,\mathcal{F}_1^{(1)}(\bar x)\Big]\delta_{ik}\delta_{j\ell}\delta_{i'j'}\delta_{k'\ell'}+$$

$$+\Big[\mathcal{F}_2^{(0)}(x)\,\mathcal{F}_2^{(0)}(\bar x)+h\,\mathcal{F}_2^{(1)}(x)\,\mathcal{F}_2^{(1)}(\bar x)\Big]\delta_{ik}\delta_{j\ell}\delta_{i'k'}\delta_{j'\ell'}\Big\} \qquad (2.47)$$

We compare with previous results and verify that it corresponds to the replacement $g_{ij} \sim \mu' \psi^+_{1i}\psi_{2j}$, with crossing obeyed by the composite field, a requirement that fixes the value of h to be given by (2.42e).

3. THE THIRRING MODEL AND STRINGS

3.1 ABELIAN SYMMETRY

As a preliminary we consider a one dimensional compactification, namely suppose that one coordinate field operator $X(z,\bar z)$ is compactified on a circle of radius R. The mode expansion is given by

$$X(z,\bar z) = X(z) + \bar X(\bar z) \qquad (3.1a)$$

$$X(z) = x^{(0)} + \frac{i}{2} p \ln z + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_{-n} z^{-n} \qquad (3.1b)$$

$$\bar X(\bar z) = \bar x^{(0)} - \frac{i}{2} p \ln \bar z + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde\alpha_{-n} \bar z^{-n} \qquad (3.1c)$$

and we define as previously

$$\tilde X(z,\bar z) = X(z) - \bar X(\bar z) \qquad (3.1d)$$

with currents

$$J(z) = \frac{i}{\sqrt\pi}\frac{\partial X}{\partial z} \qquad (3.1e)$$

$$\bar J(\bar z) = \frac{-i}{\sqrt\pi}\frac{\partial \bar X}{\partial \bar z} \qquad (3.1f)$$

The energy momentum tensor is given by

$$T(z) = 2\pi \; : J(z)^2 : \tag{3.2a}$$

$$\bar{T}(\bar{z}) = 2\pi \; : \bar{J}(\bar{z})^2 : \tag{3.2b}$$

The associated fermionic field theory is defined by the field operator

$$\psi_{\alpha,\beta}(z,\bar{z}) = \; : e^{i(\alpha X_L(z) + \beta \bar{X}_L(\bar{z}))} :$$

$$= \; : e^{i\alpha \frac{X+\tilde{X}}{2} + i\beta \frac{X-\tilde{X}}{2}} : \tag{3.3}$$

We have the following operator product expansions (OPE's):

$$T(z)\psi_{\alpha,\beta}(\omega,\bar{\omega}) = \frac{\pi\alpha}{z-\omega} : J(z)\psi_{\alpha,\beta}(\omega,\bar{\omega}) : - \frac{i\alpha^2}{4(z-\omega)^2}\psi_{\alpha,\beta}(\omega,\bar{\omega}) \tag{3.4a}$$

$$\bar{T}(\bar{z})\psi_{\alpha,\beta}(\omega,\bar{\omega}) = \frac{\pi\beta}{\bar{z}-\bar{\omega}} : \bar{J}(\bar{z})\psi_{\alpha,\beta}(\omega,\bar{\omega}) : - \frac{i\beta^2}{4(\bar{z}-\bar{\omega})^2}\psi_{\alpha,\beta}(\omega,\bar{\omega}) \tag{3.4b}$$

Since $T(z) - \bar{T}(\bar{z})$ generates Lorentz transformations,

we readily compute the Lorentz-spin of the field $\psi_{\alpha,\beta}$

$$S = \frac{\lambda}{2} = \frac{\alpha^2 - \beta^2}{8} \tag{3.5}$$

The constant β corresponds to $g/\sqrt{\pi}$ in the Thirring model with coupling constant g.

We use now the identifications (1.23) and (1.28) in (3.3) to obtain the transformations of the field $\psi_{\alpha,\beta}$:

$$\psi_{\alpha,\beta} \longrightarrow \psi_{\alpha,\beta}\, e^{i\pi R(\alpha+\beta)} \tag{3.6}$$

under (1.23), and

$$\psi_{\alpha\beta} \longrightarrow \psi_{\alpha\beta}\, e^{\frac{i\pi}{2R}(\alpha-\beta)} \tag{3.7}$$

This tranformations correspond, in the string language, to modular transformations. Modular invariance of the theory requires that well defined operators be those invariant under the above transformations. Therefore, in general, we are required to study products the above operators. In case we have a bound state of F ψ's, we require, at the same time that both following equations be fulfilled

$$FR(\alpha+\beta) = 2n \tag{3.8a}$$

$$F\,\frac{\alpha-\beta}{2R} = 2m \qquad (3.8b)$$

n and m integers, such that the spin is

$$S = \frac{\alpha^2-\beta^2}{8} = \frac{mn}{F^2} \qquad (3.9)$$

and is, thus required to be a rational number, unless we take an infinite number of ψ's to build bound states.

Two simple minded examples are the free field case, $\beta=0$, $S=\alpha^2/8$, where $S=1$ and compactification radius $R=\sqrt{2}$ ensure invariance of ψ itself. Or else, $S=1/2$, $R=1$ requires bound states of 2 ψ's .

These results are readily generalized to a symmetry $[U(1)]^{d(7)}$. In this case we come back to Minkowski space, recalling expressions (1.9) and (1.21-1.28), implying the quantization of momentum

$$P_0^a = M_\mu\,\tilde{E}^{a\mu} \qquad (3.10)$$

$$\tilde{P}_0^a = 2L^\mu E^a{}_\mu \qquad (3.11)$$

where M_μ and L^μ are integers.

The corresponding fermionic model is described by the action

$$S = \frac{1}{\pi}\int d\sigma d\tau\Big[i\,\psi_1^{i+}\partial^-\psi_1^i + i\,\psi_2^{i+}\partial^+\psi_2^i + $$
$$+ H_{ij}\,\psi_2^{i+}\psi_2^i\,\psi_1^{j+}\psi_1^j\Big] \qquad (3.12)$$

where

$$H_{ij} = F_{ai}\,K_{aj}$$

and the spinor is $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

The formal field equations are

$$\partial^+\psi_2 = -2\pi i\,F_{ai}\,J_+^a\psi_2^i \qquad (3.13a)$$

$$\partial^-\psi_1 = -2\pi i\,K_{ai}\,J_-^a\psi_1^i \qquad (3.13b)$$

The commutation relations are given by the expressions

$$\left[J_+^a(x),\,\psi_2^i(y)\right] = -\frac{1}{2}A^{ai}\psi_2^i(y)\,\delta(x_+-y_+) \qquad (3.14a)$$

$$\left[J_-^a(x),\,\psi_2^i(y)\right] = -\frac{1}{2}B^{ai}\psi_2^i(y)\,\delta(x_--y_-) \qquad (3.14b)$$

$$[J_+^a(x), \psi_1^i(y)] = -\frac{1}{2} D^{ai} \psi_1^i(y)\, \delta(x_+ - y_+) \tag{3.14c}$$

$$[J_-^a(x), \psi_1^i(y)] = -\frac{1}{2} C^{ai} \psi_1^i(y)\, \delta(x_- - y_-) \tag{3.14d}$$

(for free fields - or critical Thirring coupling, B=D=0).

We suppose the energy momentum tensor to be of the Sugawara form. Therefore, using the above we may compute the Lorentz spin as done in (3.4), (3.5):

$$S = \frac{\lambda}{2} = \frac{1}{8}\left(\sum_a (A^{ai})^2 - \sum_a (B^{ai})^2 \right) = \frac{1}{8}\left(\sum_a (C^{ai})^2 - \sum_a (D^{ai})^2 \right) \tag{3.15}$$

A possible solution may be expressed in terms of a matrix Z^{ai} together with its inverse $(Z^{-1})^{ai} = \tilde{Z}^{ai}$, and an antisymmetric matrix $Y^{ab} = -Y^{ba}$ (other solutions exist)[7]:

$$A^{ai} = \sqrt{\lambda}\left(Z^{ai} + \tilde{Z}^{ai} - Y^{ab} Z^{bi} \right) \tag{3.16a}$$

$$B^{ai} = \sqrt{\lambda}\left(Z^{ai} - \tilde{Z}^{ai} + Y^{ab} Z^{bi} \right) \tag{3.16b}$$

$$C^{ai} = \sqrt{\lambda}\left(Z^{ai} + \tilde{Z}^{ai} + Y^{ab} Z^{bi} \right) \tag{3.16c}$$

$$D^{ai} = \sqrt{\lambda}\left(Z^{ai} - \tilde{Z}^{ai} - Y^{ab} Z^{bi} \right) \tag{3.16d}$$

The bosonized realization of the $[U(1)]^d$ spinor fields is

$$\psi_1^i = :e^{i C^{ai} X_-^a(x_-) - i D^{ai} X_+^a(x_+)}: \tag{3.17a}$$

$$\psi_2^i = :e^{-i A^{ai} X_+^a(x_+) + i B^{ai} X_-^a(x_-)}: \tag{3.17b}$$

Thus, under shifts

$$\Delta X_0^a = 2\pi E_\mu^a \tag{3.18a}$$

$$\Delta \tilde{X}_0^a = \pi \tilde{E}_\mu^a \tag{3.18b}$$

we have the transformations

$$\psi_1^i \to \psi_1^i\, e^{2\pi i \sqrt{\lambda}\left(\tilde{Z}^{ai} + Y^{ab} Z^{bi} - B^{ab} Z^{bi} \right) E_\mu^a} \tag{3.19a}$$

$$\psi_2^i \to \psi_2^i\, e^{-2\pi i \sqrt{\lambda}\left(\tilde{Z}^{ai} - Y^{ab} Z^{bi} + B^{ab} Z^{bi} \right) E_\mu^a} \tag{3.19b}$$

$$\psi_{1,2}^i \to \psi_{1,2}^i\, e^{-i\pi Z^{ai} \tilde{E}_\mu^a \sqrt{\lambda}} \tag{3.19c}$$

As in the previous case, bound states must be considered.

3.2. NON ABELIAN SYMMETRY

In the non abelian Thirring model, a complete operator solution is not known, but there are some helpful expressions, which may be computed, and used to bound state calculations, and to fix the spin of the field.

For the product spinor-antispinor we have [10]

$$\psi_1^i(x)\,\psi_1^{j\dagger}(y) = \mathbb{C}\,(x_+ - y_+)^{-A}(x_- - y_-)^{-B} \times$$

$$\times\, e^{\frac{-i}{2C_0}\left[(a+\bar{a})\int_{x_+}^{y_+} j_+(\omega_+) + (a-\bar{a})\int_{x_-}^{y_-} j_-(\omega_-)\,d\omega_-\right]} M^{ij}(x,y) \qquad (3.20)$$

where M satisfies

$$\partial^+ M = 0$$

$$\partial^- M = \frac{-1}{n+k}\left\{\frac{1}{2}\lambda^a : j_-^a M : + \frac{1}{4\pi}\frac{1}{i(x_- - y_-)+\epsilon}\left[\lambda^b M \lambda^b - \frac{2(n^2-1)}{n} M\right]\right\} \qquad (3.21)$$

with the condition

$$M(x,x) = \mathbb{1} \qquad (3.22)$$

Thus

$$M(x,y) \sim e^{iC'\int_x^y \lambda^b y^b d\omega} \sim e^{iC\lambda^b\left(X^b_{(x)} - X^b_{(y)}\right)} \qquad (3.23)$$

Again, we have non abelian fermionization and abelian bosonization formulae, which are the same as the usual (2.46).

Comparing abelian and non abelian cases, defined on the same compactification torus , we have the identifications

$$H_{ij} \longleftrightarrow (\tau^a)_{ii}\,(\tau^a)_{jj} \qquad (3.24a)$$

or

$$F_{ai} \leftrightarrow \tau^a{}_{ii} \quad ; \quad K_{ai} \leftrightarrow \tau^a{}_{ii} \qquad (3.24b)$$

Also

$$A^{ai} \leftrightarrow (\tau^a)^{ii} \qquad (3.24c)$$

$$C^{ai} \leftrightarrow (\tau^a)^{ii} \qquad (3.24d)$$

for an even self dual lattice [17], Ψ is modular invariant, and there are no further constraints in the non abelian piece. Only abelian pieces are arbitrary.

4. VÉRTICES AND STRING THEORY

Vertices in compactified string theory have been discussed in detail by Gepner and Witten[18]. As it turns out, a vertice is a product of the Minkowski space vertex and the compactified piece. Let us discuss the simplest case of the tachyon. The product

$$V(z,\bar{z}) = \phi^{R}_{comp.} : e^{i p^{\mu} X_{\mu}(z)} :$$

where $X_{\mu}(z)$, $\mu = 0, \cdots, D-1$ are the uncompactified variables, p_{μ} the momentum and $\phi^{R}_{comp.}$ is a representation of the group acting on the compactified manifold. The latter is also an element of the Verma module corresponding to the Kac-Moody algebra. Thus it may be represented by the WZW field g_{ij}. Or else, since it is of the form $\exp i K^a X^a(z,\bar{z})$, it may be well described by (3.23), namely a bound state of the previously defined fermion, or simply an expression as (3.17), which is the own fermion. The only requirement is that of modular invariance, as we discussed previously. Correlators have a product form, of the Minkowski piece by the compactified part. Consider as an example a bound state

$$j^{ab}(\xi) = N\left[\psi^a(\xi)\,\psi^{b+}(\xi)\right] \tag{4.1}$$

We have an explicit formula for

$$\left\langle \psi^a(\xi+\epsilon)\,\psi^{b+}(\xi)\,\psi^{c+}(\xi')\,\psi^d(\xi'+\epsilon')\right\rangle \tag{4.2}$$

given by (2.38). We compute the above for $\epsilon, \epsilon' \to 0$, using

$$F\left(-\frac{1}{\chi}, \frac{1}{\chi}, 1+\frac{n}{\chi}; x\right) \simeq 1 - \frac{x}{\chi(n+\chi)} \tag{4.3}$$

$$\chi = n + k$$

In the above limit we have

$$\left\langle \psi^a(\xi+\epsilon)\,\psi^{b+}(\xi)\,\psi^{c+}(\xi')\,\psi^d(\xi'+\epsilon')\right\rangle =$$

$$\delta^{ab}\delta^{cd}(\epsilon\epsilon')^{-2\Delta} + h\left[\frac{\epsilon\epsilon'}{(\xi-\xi')^2}\right]^{\Delta_1-2\Delta}\left(\delta^{ab}\delta^{cd} - n\delta^{ac}\delta^{bd}\right)$$

$$-\left(\delta^{ab}\delta^{cd}\frac{k-n}{nk(n+k)} - k\delta^{ac}\delta^{bd}\right)\frac{(\epsilon\epsilon')^{1-2\Delta}}{(\xi-\xi')^2} + \cdots \tag{4.4}$$

The first contribution is trivial, and should be subtracted. For $k \neq 1$, the second contribution is the only one remaining after renormalization is performed. We have

$$\left\langle N(\psi^a\psi^{b+})(\xi)\,N(\psi^{c+}\psi^d)(\xi')\right\rangle =$$

$$= \frac{h\,\mu^{-8\Delta}}{(\xi-\xi')^{\frac{2}{n(n+k)}}}\left(\delta^{ab}\delta^{cd} - n\,\delta^{ac}\delta^{bd}\right) \tag{4.5}$$

Therefore we have for $j^{ab} = N(\psi^a \psi^{b+})$ an anomalous dimension

$$\gamma_j = \frac{1}{n\,(n+k)} \qquad (4.6)$$

For $k=1$, we have $h=0$, and the dimension j of is canonical, $\gamma_j = 1$. In this case we have

$$\langle j^{ab}(\xi)\, j^{dc}(\xi')\rangle = \frac{\mu^{-8\Delta}}{(\xi-\xi')^2}\left(\delta^{ab}\frac{1-n}{n(n+1)} - \delta^{ac}\delta^{bd}\right) \qquad (4.7)$$

Therefore the problem is non trivial for $k \neq 1$. The case $k=1$ has the values of free field theory for the dimensions.

5. CONCLUSIONS

We analysed the issue of equivalence between bosons and fermions at the level of Green functions, concluding that the non abelian Thirring model at critical coupling presents as a defining field a representation of the conformal algebra, whose bound state is the bosonic WZW field, the level of both representatives being the same (k).

Therefore, using this result we may study vertex operators of compactified bosonic string theories, which turns out to be the elementary field operator in the fermionic language. Thus, a fermion operator $\psi \sim e^{i\alpha x}$ of spin $\alpha^2/4$ corresponds to a vertex operator of momentum α. Bound states of ψ obeying modular invariance can be computed, and in the case $k \neq 1$, anomalous dimensions arise naturally, as discussed in the last section.

At last, in the non abelian theory, the number of free parameters is very much reduced, contrary to the abelian case, where, compactification radii are completely uncorrelated. The non abelian symmetry group, being connected correlates all radii, and the only freedom left is in the abelian piece. This property can have some non trivial role in further developments.

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