UNIVERSIDADE DE SÃO PAULO

PUBLICAÇÕES

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
BRASIL

IFUSP/P - 711

THE SCHWINGER FUNCTIONS OF A RATIONAL INTERACTION: LOCAL EXISTENCE OF THE BOREL TRANSFORM

G.A.F.T. da Costa
Instituto de Física Teórica
Rua Pamplona, 145
01405 São Paulo, SP, Brazil

M. Gomes
Instituto de Física, Universidade de São Paulo

Maio/1988

THE SCHWINGER FUNCTIONS OF A RATIONAL INTERACTION: LOCAL EXISTENCE OF THE BOREL TRANSFORM

G. A. F. T. da Costa[1]

Instituto de Física Teórica
Rua Pamplona, 145
CEP 01405 - São Paulo - SP, Brazil

M. Gomes[2]

Instituto de Física da USP - Departamento de Física Matemática
C. P. 20516 - São Paulo, S.P., Brazil

Abstract

We prove the local existence of the Borel transform of a two dimensional field theoretical model characterized by the rational interaction $\frac{g^2x^6}{1+gx^2}$.

This work was partially supported by FAPESP (1) and CNPq (2).

In a recent paper, [1], we proved the Borel summability of the perturbative series for the energies of the quantum mechanical system characterized by the non-polynomial interaction

$$\frac{g^2x^6}{1+gx^2} \tag{1}$$

The proof of Borel summability was done by establishing that the perturbative spectrum is strongly asymptotic and then applying a modified version of Nevanlinna's theorem [2]. Here we want to extend these studies considering (1) as the self interaction of a quantum field. To achieve renormalizability, due to the nonpolynomial character of the interaction (1), we shall restrict our analysis to two dimensions. We are then able to prove the local existence of the Borel transform of the Schwinger functions of the Euclidean theory. This is a first step towards a complete proof of Borel summability.

The physical motivation for considering the interaction (1) comes from laser theory models where the reduction of the Fokker-Planck to the Schrodinger equation produces interactions similar to the above one (see for example [3] and the references mentioned therein) . Besides that, the study of (1) has its own merits for, as it is a rational function, the perturbative series is singular both due to the bad behaviour at x large and to the poles occurring in the denominator of the potential.

Basically, there are two reasons why in general the perturbative series is divergent. Firstly, the number of diagrams can grow too fastly (tipically with $n!$) with the order n of perturbation and, secondly, some individual diagrams dominate giving too large contributions. The second reason is peculiar to renormalizable theories whereas the first phenomena occurr also in superrenormalizable models like ours. We will have to find therefore bounds in the number of diagrams which contribute at a given order. An important result concerning this is the following Lemma.

Lemma 1. *The number, $\gamma(n)$, of connected diagrams with V vertices contributing to order n to the E point Schwinger function of the two dimensional model with Lagrangian*

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{g^2\varphi^6}{1+g\varphi^2} \tag{2}$$

satisfies the inequality

$$\gamma(n) \leq 2^n(n!)(V!)n^E \tag{3}$$

To prove (3), is convenient to collect some combinatoric relations valid for a generic connected graph G of our model.

Lemma 2. *In a graph G contributing to the Schwinger functions of the model (2), the following relations are valid:*

1) $L = I - V + 1$.

2) $n = \frac{1}{2}\sum kV_k - V$.

3) $n = I + \frac{E}{2} - V$.

4) $V \leq \frac{n}{2}$.

5) $n = L - 1 + \frac{E}{2}$. *Thus* $n = L$ *if* $E = 2$

6) $L < \frac{3}{2}n - \frac{E}{2} + 2$.

7) $I < \frac{3}{2}n - \frac{E}{2} + 1 < \frac{3}{2}n + 1$.

8) $I < \frac{3}{2}n$ *if* $E = 2$.

9) $\sum kV_k = E + 2I$.

10) $6V \leq \sum kV_k \leq 3n$.

where L = number of loops; E = number of external lines; I = number of internal lines; V = number of vertices.

1) is just Euler's relation expressing the number of loops of a diagram in terms of the number of internal lines and vertices. To verify the other relations, we use

$$\frac{g^2\varphi^6}{1+g\varphi^2} = \sum_{\nu=2}(-1)^\nu g^\nu \varphi^{2\nu+2} \tag{4}$$

A k-vertex is a vertex in which k lines are met. From (4) a k-vertex has a factor $g^{k/2-1}$. Thus if the graph has V_k k-vertices then the diagram will be of order $n = \sum_k(\frac{k}{2}-1)V_k = \frac{1}{2}\sum_k kV_k - V$, which proves 2). 3) is proven using 2) and the relation $\frac{1}{2}\sum kV_k = I + E/2$ obtained by counting the number of line endings in G. Now, since at a given vertex there are at least 6 lines, $n = \sum_k(\frac{k}{2}-1)V_k \leq 2\sum_k V_k = 2V$, giving 4). The other relations are proven similarly.

The counting of the number of diagrams is greatly facilitated by a method developed in ref. [4]. Let G be a proper diagram. We can always draw it so that all its vertices are on a circumference. Therefore two lines of any given vertex of G are used to link it to other two vertices. To some vertices will be also attached external lines. The number of ways of distributing E external lines in V vertices is certainly less than V^E (imagine putting three stones in four boxes).

After having distributed the external lines there will be still a number l of fields to be contracted. These contractions will produce $(l-1)!!$ graphs. The number of diagrams with V vertices has also factor $V!$ c oming from the permutations of the vertices. Putting all this together we arrive to the conclusion that the number of diagrams with V vertices is bounded by

$$\beta = V!V^E (l-1)!! \tag{5}$$

The number l can be written in terms of n and E, $l = 2n - E$. Indeed, from its definition $l = 2I - 2V = 2n - E$, where 3) in the Lemma 2 has been used. Then

$$\beta = V!V^E 2^{n-\frac{E}{2}}(n - \frac{E}{2})! \leq V!V^E 2^n n! \tag{6}$$

This number bounds the number of graphs having a fixed, $\{V_k; k \geq 6\}$, configuration of vertices. The result (3) follows from relation 4) of lemma 2.

The perturbative expansion for the Schwinger functions are obtained from (1) by expanding it in powers of g. The only divergences that are found in this process are associated to tadpole diagrams (graphs having just one loop and one internal line) which are removed by Wick ordering the interaction Lagrangian with respect to the mass m. In this situation it can be proven, [5], that in order n individual amplitudes are bounded, i. e.,

$$A_n \leq K'^n \tag{7}$$

where K' is a constant independent of the topology of the associated diagram. Now, the order n total amplitude, $G_n^{(E)}$, is defined as

$$G_n^{(E)} = (-1)^n \sum{}'(-1)^V \frac{A_{n,\{V_k\}}}{\prod_k V_k!} \tag{8}$$

where the sum $\sum'$ is over all possible assignements $\{V_k\}; k \geq 6$ to the vertices such that $\sum(\frac{k}{2} - 1)V_k = n$. $A_{n,\{V_k\}}$ is the sum of all possible graphs corresponding to a given assignement. It can be easily seen that, in a given order n, there are 2^n configurations satisfying $\sum(\frac{k}{2} - 1)V_k = n$.

From (7), (8) and (3) we have therefore that the order n total amplitude is bounded by

$$4^n n^E n! \, sup[\frac{V!}{\prod_k V_k!}]K'^n \tag{9}$$

where the supremum is to be taken over all configurations satisfying $\sum(\frac{k}{2} - 1)V_k = n$. This supremum is bounded by a positive constant to the power n. We have therefore

$$G_n^{(E)} \leq 4^n (n!) n^{E+1} K^n \tag{10}$$

This equation implies that the perturbative series has a Borel transform free of singularities within a ball of radius $\frac{1}{K}$ with center at the origin of the Borel plane.

We can now enunciate our main result.

Theorem. *The expansion*

$$\sum B_n b^n , \qquad B_n = \frac{G_n^{(E)}}{n!} \tag{11}$$

for the Borel transform of the E point Schwinger functions of the theory (2) converges within a circle of radius $1/K$ where it defines an analytic function of b.

To go further, having proved the local existence of the Borel transform, we need now to extend its domain of analiticity to a neighborhood of the positive real axis in the complex Borel plane. Work in this direction is in progress.

Acknowledgment. The authors wish to thank V. Rivasseau for helpful discussions and a critical reading of the manuscript.

References

1. G. A. T. F. da Costa and M. Gomes, " A Non Polynomial Interaction: Stability and Borel Summability", preprint, Instituto de Fisica Teorica IFT/P-11/88.
2. A. Sokal, J. Math. Phys. 21, 2, 1980.
3. H.Risken, H. D. Vollmer , Z. Phys. 201, 323, 1967.
   H. Haken: Laser theory. In Encyclopedia of Physics, Vol. XXV/2c. Princeton N.J.: Van Nostrand, 1970.
4. A. Jaffe, Comm. Math. Phys. 1, 127, 1965.
5. J. Feldman, J. Magnen, V. Rivasseau and R. Seneor, Comm. Math. Phys. 98, 273, 1985.