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GEOMETRIC PHASES AND GRAVITATIONAL ANOMALIES

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ABSTRACT

An analysis is made of the motion of a quantum particle in a gravitational field, and it is shown that a geometric phase arises under certain conditions. The connection of this result with the emergence of gravitational anomalies is described.

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1. INTRODUCTION

Classical symmetries which do not exist at all at the quantum-mechanical level occur naturally in field theory, but are rare in systems with a finite number of degrees of freedom. These "anomalous symmetries" are always broken, because the introduction of a regulator, indispensable for renormalization, either spoils some symmetry or violates some other requisite, such as unitarity. Anomalies appear under a variety of guises [1]: gauge and non-gauge, Abelian and non-Abelian, perturbative and non-perturbative. Some, such as the axial U(1) in QCD and certain non-perturbative anomalies in string theory [4], are welcome, while others, like the axial gauge anomalies in the standard model of electroweak interactions, are truly undesirable Anomalies in the latter category render the theory inconsistent, so their absence can be turned into a powerful criterion in reducing the collection of renormalizable theories; indeed, the condition of anomaly-freedom usually fixes, or at least restricts, the value of an otherwise arbitrary parameter in the theory [1]. On the other hand, there exists a class of model theories (so far all bidimensional) that can be consistently quantized in the presence of anomalies, at the expense of gauge invariance [2] or canonical quantization rules [3]. Probably the last word has not been said yet as to when it is inevitable to free oneself from anomalies.

In view of this prominent role played by anomalies in quantum field theory, it is essential to understand them from a variety of viewpoints. Perhaps the most subtle and least explored are the anomalies which arise in theories with gravitational couplings. These (gravitational) anomalies appear in two versions - perturbative [4] and global [5] - obstructing the realization of spacetime symmetries. In refs. 3 and 4 a number of interesting results concerning gravitational anomalies were obtained with the Lagrangian (or path-integral) formalism. In this approach one examines the effective interaction induced by integration over the fermionic sector, looking for explicit symmetry breaking or ambiguities under global transformations.

An alternative approach, applied to chiral anomalies [5], is the Hamiltonian or Hilbert space formulation. In this language an anomaly occurs whenever the symmetry cannot be implemented as a true representation, but only as a projective, or ray representation; such cases involve topologically unremovable phases, which break the gauge invariance. Here, when one examines the effective Hamiltonian for the bosonic fields one finds [6,7] that the anomalous content of the theory is concisely comprised in a phase whose origin is purely geometric: the phase acquired by the wavefunction of a system subject to (an adiabatic) periodic potential [8].

In this note a study is begun of gravitational anomalies which makes use of the Hamiltonian interpretation. It is shown that a quantum mechanical particle with spin and interacting only with a gravitational field can develop a geometric phase. The next section reviews Berry's results [8] on adiabatic

phases and a generalization due to Aharonov and Anandam [9]. In the following section an analysis is made of the motion of a spinning particle in curved space. Some other related issues are touched upon in the concluding section.

2. GEOMETRIC PHASES

Let us review how a geometric phase arises in quantum mechanics, beginning with the case of adiabatic evolutions. Consider a system suhect to a "slowly" changing external field $\vec{Q}(t)$, so that it is meaningful to describe the state of the system in terms of the eigenfunctions of the "momentary" Hamiltonian, defined by

$$H(Q(t))|\psi_n(Q(t))\rangle = E_n(Q(t))|\psi_n(Q(t))\rangle$$
$$\langle\psi_n(Q(t))|\psi_m(Q(t))\rangle = \delta_{mn} \quad . \tag{2.1}$$

As first pointed out by Berry [8], the Adiabatic theorem [10] is modified if the motion of the background field $\vec{Q}(t)$ is periodic. The conventional version of the theorem asserts that if the initial state vector is $|\psi_n(Q(0))\rangle$, then at a later time t it will have evolved into

$$|\psi_n(Q(t))\rangle\, e^{i\varphi_n(t)}$$

where $$\varphi_n = -\int_0^t dt'\, E_n(Q(t')) \quad ,$$

provided the background changes slowly enough(*) that transitions do not take place. The phase φ_n is called dynamical because it depends on the functional form of H. In words, the system remains in the "same" eigenstate labelled by $E_n(Q(t))$, riding along with the background.

However, if the background field resumes its initial configuration after a time T, i.e., $\vec{Q}(T) = \vec{Q}(0)$, the state vector is instead

$$|\psi_n(Q(0))\rangle\, e^{i\varphi_n + i\gamma_n} \quad ,$$

where the extra phase γ_n, differently from φ_n, depends only on the geometry of the surface $E_n(Q)$ in Q-space [8]. From the time-dependent Schrödinger equation, one finds

$$\gamma_n = \int_0^T dt\, \langle\psi_n(Q(t))|i\frac{d}{dt}|\psi_n(Q(t))\rangle$$

$$= \oint_C d\vec{Q}\cdot\langle\psi_n(Q)|i\vec{\nabla}_Q|\psi_n(Q)\rangle \qquad (2.2)$$

where C is the closed circuit in Q-space generated by the

(*) Or rather $\Delta H = \dfrac{|\langle\psi_n|\frac{\partial H}{\partial t}|\psi_m\rangle|}{E_m - E_n} \ll E_m - E_n$.

background motion. The last expression shows that γ is a functional of C, i.e., independent of the parametrization t.

When the circuit is open, it is possible to absorb the extra phase in ψ_n, since eqs. (2.1) fix it only up to a phase factor; this is no longer possible once the circuit closes, because γ is non-integrable.

From the viewpoint of parameter space this means there is a connection(§)

$$\vec{A}_n(Q) = \langle\psi_n(Q)|i\vec{\nabla}_Q|\psi_n(Q)\rangle \quad ,$$

in terms of which

$$\gamma_n[C] = \oint_C d\vec{Q}\, \vec{A}_n(Q) \quad .$$

An alternative expression follows from (2.1) and (2.2):

$$\gamma_n = -\int_S d\vec{S}\cdot \mathrm{Im} \sum_{m\neq n} \frac{\langle\psi_n|\vec{\nabla}_Q H|\psi_m\rangle \times \langle\psi_m|\vec{\nabla}_Q H|\psi_n\rangle}{[E_m(Q) - E_n(Q)]^2} \quad .$$

The denominator provides a hint as to why a degeneracy in Q-space is a common feature of a large number of systems with non-vanishing γ_n.

(§) The phase change $|\psi_n\rangle \to e^{i\chi(Q)}|\psi_n\rangle$ induces $A_n \to A_n - \nabla_Q\chi$. $\vec{A}$ becomes non-Abelian if level n is degenerate [11].

The paradigm of such systems is a degenerate two-level system subject to an interaction linear in the external field, namely $H_{int} \propto \vec{\sigma}\cdot\vec{Q}(t)$. This class is very large indeed, including certain triatomic molecules, the quantum Hall effects, chiral anomalies, and Skyrmions [12]. Perhaps its simplest version is a spin $\frac{1}{2}$ particle in a cyclic magnetic field. See figure 1. (Here, $\vec{Q}(t)$ is the magnetic field vector.)

For this class of system, it can be shown that $\gamma_n[C]$ is proportional to the solid angle Ω subtended by C from the origin $Q=0$ (Figure 1). Since γ is the flux of $\vec{A}$ through C, this means there is a monopolar "magnetic" flux in parameter (Q) space, as if there were a monopole sitting at the origin. In particular, γ is unchanged by circuit deformations which preserve the solid angle. This property and the fact that γ vanishes for an open circuit are what make it a topological quantity.

In principle, the only restriction on the value of γ is that it must be a real number (this follows from $\langle\psi_n|\psi_n\rangle = 1$). Under certain conditions even a system made out of bosons can have $\gamma=\pi$, in which case its wave function changes sign upon a full rotation in coordinate space. Perhaps the most familiar system exhibiting this property is the Skyrmion - a soliton in a theory of purely bosonic fields, which is nevertheless a fermion [12].

We conclude this summary on the adiabatic phase with a brief remark on the connection between geometric phases and

chiral anomalies. As mentioned in the introductory section, the integration over the fermionic sector of an anomalous theory induces in the bosonic sector an interaction proportional to the phase γ. It turns out that the origin of this phase is very much the same as in the example just described; it is a property of the degenerate fermionic vacuum in the presence of a bosonic background, which can be attributed to a magnetic flux in parameter space. Since the parameters here are the bosonic fields, the anomalous term will describe monopoles living in the configuration space of bosonic fields. But this is precisely what Wess-Zumino terms are [13]. The geometric phase thus provides an interesting stand point on anomalies, tracing it in the end to the fact that energy eigenfucntions acquire phases during closed loop excursions in parameter space, or rather they form twisted line bundles over S^1 [6].

Now let us free the concept of the geometric phase from the restriction to adiabatic evolutions, and consider motions of the parameter $\vec{Q}$ that are just periodic [9]. Let $\psi(t)$ be a normalized state vector in the Hilbert space H evolving with the Schrödinger equation $i\frac{\partial\psi(t)}{\partial t} = H(t)\psi(t)$. Suppose further that the motion is periodic, i.e., $\vec{Q}(T)=\vec{Q}(0)$, so that

$$\psi(T) = e^{i\Lambda}\psi(0) \quad , \tag{2.3}$$

Λ real.

Consider now the space of <u>rays</u> P_H associated with

$\mathcal{H}$, obtained by identification of all vectors which differ by a complex number. In order to express the results in a concise form it is convenient to define a mapping π which projects a vector ψ onto its ray $\hat{\psi}$, namely

$$\pi(\psi) = \{\hat{\psi} : \psi = z\hat{\psi}\ , z \in \mathbb{C}\} \quad ,$$

so π projects $\mathcal{H}$ onto $P_{\mathcal{H}}$, $\pi(\mathcal{H}) = P_{\mathcal{H}}$. See figure 2.

Notice that each periodic evolution defines a sequence of states in $\mathcal{H}$ – a curve $\Gamma : (0,T) \to \mathcal{H}$ – that begins and ends on ray $\hat{\psi}$. Through the mapping π it defines also a closed curve $\hat{\Gamma}$ in $P_{\mathcal{H}}$, namely the projection $\pi(\Gamma) = \hat{\Gamma}$. See figure 2.

One defines the geometric phase β as that piece of the total phase Λ which is not dynamical:

$$\beta = \Lambda + \int_0^T dt\, \langle\psi(t)|H|\psi(t)\rangle \quad .$$

Although β may seem to depend on the circuit Γ, it is a function of $\hat{\Gamma}$ only. In other words, β is the same for all curves Γ in $\mathcal{H}$ which project onto the same curve $\hat{\Gamma} = \pi(\Gamma)$ in $P_{\mathcal{H}}$. This assertion can be easily proved in two short steps.

First express $\beta(\Gamma)$ in terms of $\tilde{\psi}(t) = e^{if(t)}\psi(t)$, with f so chosen that $f(T) - f(0) = \Lambda$, to find

$$\beta(\Gamma) = \int_0^T dt\, \langle\tilde{\psi}|\, i\frac{d}{dt}\, |\tilde{\psi}\rangle \quad .$$

Next we consider another curve Γ' such that $\pi(\Gamma') = \pi(\Gamma) = \hat{\Gamma}$. This means that along Γ' there exists a phase $\alpha(t)$ such that

$$\psi'(t) = e^{i\alpha(t)}\psi(t) = e^{i\alpha(t)+if(t)}\, \tilde{\psi}(t) \quad .$$

Now choosing $\alpha(t)$ so that $\alpha(T) = \alpha(0)$, we have

$$\psi'(t) = e^{if'}\hat{\psi}(t) \qquad \text{with} \qquad f'(T) - f'(0) = \Lambda \quad ,$$

and

$$\beta(\Gamma') = \phi + \int_0^T dt\, \langle\psi'(t)|H|\psi'(t)\rangle = \int_0^T dt\, \langle\tilde{\psi}|\, i\frac{d}{dt}|\tilde{\psi}\rangle = \beta(\Gamma) \quad .$$

Hence, $\beta[\hat{\Gamma}]$ is independent of the Hamiltonian H, the phase Λ, and the parametrization t; it is a geometric property of closed loops in $P_{\mathcal{H}}$. An explicit calculation in the next section will substantiate this feature of β.

Not with standing this distinction, it is easy to verify that β reduces to γ when adiabatic conditions prevail [9]. Moreover, the β associated with a given evolution can be calculated with the formula for γ in terms of eigenfunctions, since an adiabatic Hamiltonian can always be found that generates the given motion. As our objective is just to show that a non-vanishing phase arises, the phase β will prove more convenient.

3. SPINNING PARTICLE IN A GRAVITATIONAL FIELD

As will become clear soon, it is interesting to consider the motion of a particle (with non-vanishing angular momentum subject to a stationary gravitational field, that is to say, one described by a metric which is x^o (time)-independent, and an interval which is not invariant under time reversal ($x^o \to -x^o$).

The spin vector S_μ of a particle in free fall obeys the equations [14]

$$\left(\frac{d}{d\tau} - \Gamma^\alpha{}_{\beta\nu} \frac{dx^\nu}{d\tau} g^{\mu\beta}\right) S_\mu = 0 \quad .$$

It is further constrained by the condition $S^\alpha \frac{dx^\alpha}{d\tau} = 0$, expressing the fact that only three of its components are independent. [In the particle's rest frame, $S_\mu = (\vec{S}, 0)$, so the spin is always orthogonal to the velocity $\frac{dx^\mu}{d\tau} = \gamma v^\mu$]. Using the constraint to eliminate the component S_o, we are left with

$$\frac{dS_i}{dt} = (\Gamma^j_{io} - \Gamma^o_{io} v^j + \Gamma^j_{ik} v^k - \Gamma^o_{ik} v^k v^j) S_j \quad .$$

For our purposes it suffices to solve the equations within the Post-Newtonian approximation. Accordingly, the expansion of the metric in powers of the particle velocity v, up to second order, gives [14]

$$\frac{d\vec{S}}{dt} = \left[\frac{1}{2} (\vec{\nabla} \times \vec{\zeta}) \times + \frac{\partial \phi}{\partial t} + (\vec{v} \cdot \vec{\nabla})\phi + 2\vec{\nabla}\phi(\vec{v} \cdot \ ) - \vec{v}(\vec{\nabla}\phi \cdot \ )\right]\vec{S} \tag{3.1}$$

where $\zeta_i = O(v^3)$ and $\phi = O(v^2)$ are defined by the equations (*)

$$g_{io} = \zeta_i + O(v^5) \quad , \quad g_{oo} = -1 - 2\phi + O(v^4) \quad .$$

At this point one notices that although the magnitude of $\vec{S}$ is not constant, there is another vector whose motion is a pure precession. Since the constant value of $S^\alpha S_\alpha$ is proportional to

$$(1+2\phi)\vec{S}^2 - (\vec{v} \cdot \vec{S})^2$$

one defines a new spin vector

$$\vec{\mathcal{S}} = (1+\phi)\vec{S} - \frac{1}{2} (\vec{v} \cdot \vec{S})\vec{S} \tag{3.2}$$

so that $\vec{\mathcal{S}}^2 = \text{constant}$.

The vector $\vec{\mathcal{S}}$ is the one that processes; differentiating (3.2), we find

$$\frac{d\vec{\mathcal{S}}}{dt} = \frac{d\vec{S}}{dt} + \left[\frac{\partial \phi}{\partial t} + \vec{v} \cdot \vec{\nabla}\phi + \frac{1}{2} \vec{\nabla}\phi(\vec{v} \cdot \ ) + \frac{1}{2} \vec{v}(\vec{\nabla}\phi \cdot \ )\right]\vec{S} \quad ,$$

(*) For a sphere of mass M and angular momentum J at rest, $\phi = -GM/r$, $\vec{\zeta} = \frac{2G}{r^3} \vec{r} \times \vec{J}$.

and a comparison with (3.1) leads to

$$\frac{d}{dt} = \vec{\Omega} \times \vec{S}$$

with $$\vec{\Omega} = -\frac{1}{2}\,\vec{\nabla} \times \vec{\zeta} - \frac{3}{2}\,\vec{v} \times \vec{\nabla}\phi \quad .$$

In order to see that a geometric phase arises in this situation, it is simplest to consider first a spin $\frac{1}{2}$ particle at rest. If the gravitational field is stationary, such as the one generated by a rotating body, then ζ is not equal to zero, and there is spin precession even for a particle at rest. The Hamiltonian becomes $-c\,\vec{\Omega}.\vec{S}$ with $c = 1+\phi$. The system is therefore analogous to a spin precessing in a uniform magnetic field, and the geometric phase can be computed as in ref. 9.

For convenience, let us choose the coordinate axes so that $\vec{\Omega}$ is along the z-axis, that is, $H = -\mu\Omega\sigma_z$, with μ = const. and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If the initial state is $|\psi(0)> = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$, then

$$|\psi(t)> = e^{i\mu\Omega t\sigma_z}|\psi(0)> = \begin{pmatrix} e^{i\mu\Omega t} & \cos\theta/2 \\ e^{-i\mu\Omega t} & \sin\theta/2 \end{pmatrix} \quad .$$

The spin vector $<\psi|\vec{\sigma}|\psi>$ precesses at an angle θ abouth the z-axis with a period $T = \pi/\mu\Omega$. Insertion of $|\psi(t)>$ in the

expression for β gives

$$\beta = (1-\cos\theta)\mu\Omega t \quad .$$

After one period, $\beta = (1-\cos\theta)\pi$, thus confirming that the dynamical dependence drops out of $\beta(T)$.

Once more the phase equals half the solid angle subtended by the curve traced by a vector; this is very much like in the adiabatic case, but while there the vector was the magnetic field, here it is the spin.

When the particle is moving, the Hamiltonian becomes $-\vec{\Omega}.\vec{S}$, where $\vec{\Omega}$ depends on the position and velocity of the particle, and $\vec{S}$ is given by (3.2). As a consequence, the motion is more complex, but the spin still precesses and formula (2.3) remains valid. There is no <u>a priori</u> reason why β should vanish for an arbitrary circuit $\hat{\Gamma}$.

4. CONCLUSIONS

The result of section 3 brings new members - those with purely gravitational interactions - into the class of systems that display a geometric phase. This result is interesting also because it provides a first quantized framework for one to think about the elusive issue of anomalies. In analogy with chiral anomalies, it suggests that a quantum field theory of fermions in a gravitational field may develop anomalies - a prediction which turns out to be true under certain conditions [4,5]. In order to avoid misunderstanding we emphasize that this result does not imply symmetry breaking in the first-quantized system, which has a finite number of degrees of freedom. As pointed out in ref. 15 , the connection between geometric phases and Wess-Zumino terms in quantum mechanics is valid only under adiabatic conditions; if transitions between spin states take place, Berry phases with opposite signs add up to a vanishing net effect.

Several aspects of this work can be pursued further. It would be interesting to treat the particle motion in more generality; for this one needs a Hamiltonian description of a spinning particle in general relativity. In the case of chiral anomalies, the geometric phase can be expressed in terms of indices of Dirac operators; what happens when gravitational couplings are present? One can also investigate how the anomalous terms affect the quantization of the gauge sector [7].

Finally, it is perhaps worthwhile to call attention to a point which arises in connection with the Lense-Thirring effect of a spinning particle in the vicinity of a large rotating mass. This effect indicates that it is the whole mass of distant matter that selects, amongst all frames, those which are inertial. On the other hand, short distance effects determine whether spacetime symmetries are realized at all. Seemingly, these two disparate scales operate together - one specifying whether intertial frames exist; the other, who they are. It would be interesting to investigate this point further, looking

for a possible connection between microscopic and cosmological scales, which is otherwise expected only in the very early stages of the universe.

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REFERENCES

[1] Anomalies, Geometry, and Topology, ed. A. White (World Scientific, Singapore, 1986).
Current Algebra and Anomalies, eds. S. Treiman et al. (Princeton U.P./World Scientific, Princeton/Singapore, 1985).

[2] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54, 1219 (1985).
E. Abdalla and K. Rothe, São Paulo preprint, # IFUSP/P-641 (1987).

[3] L. Faddeev and S. Shatashvili, Phys. Lett. 167B, 225 (1986).

[4] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234, 269 (1983).
L. Alvarez-Gaumé and P. Ginsparg, Ann. Phys. 161, 423 (1985).

[5] E. Witten, Comm. Math. Phys. 100, 197 (1985).

[6] P. Nelson and L. Alvarez-Gaumé, Comm. Math. Phys. 99, 103 (1985).

[7] A. Niemi and G. Semenoff, Phys. Rev. Lett. 55, 927 (1986).

[8] M.V. Berry, Proc. Roy. Soc. London A392, 45 (1984).

[9] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593 (1987).

[10] A. Messiah, Quantum Mechanics, vol. 2 (North-Holland, Amsterdam, 1962).

[11] F. Wilczek and A. Zee, Phys. Rev. Lett. 52, 2111 (1984).

[12] I. Aitchison, Talk at the Seventh General Conference of the European Physical Society, Helsinki, Finland, August 1987; Oxford preprint.

[13] E. Witten, Nucl. Phys. B223, 422 (1983).

[14] Here we follow S. Weinberg, Gravitation and Cosmology (John Wiley and Sons, New York, 1972).
The Minkowski metric is $\eta_{\mu\nu} = (-,+,+,+)$; $x^0 \equiv t$; τ is the proper time: $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$.

[15] M. Stone, Phys. Rev. D33, 1191 (1986).

FIGURE CAPTIONS

Fig. 1 – The vector $\vec{Q}(t)$ traces a curve C in Q-space which subtends a solid angle Ω from the origin.

Fig. 2 – H is the Hilbert space of wave-functions ψ, P_H the space of rays $\hat{\psi}$, and π is the projection map $\pi(\psi)=\hat{\psi}$. Γ is a curve in H, while $\hat{\Gamma}$ is a closed curve in P_H.

Fig. 3 – The spin vector $\langle\psi(t)|\vec{\sigma}|\psi(t)\rangle$ traces a curve in R^3 which determines a closed curve $\hat{\Gamma}$ in P_H.

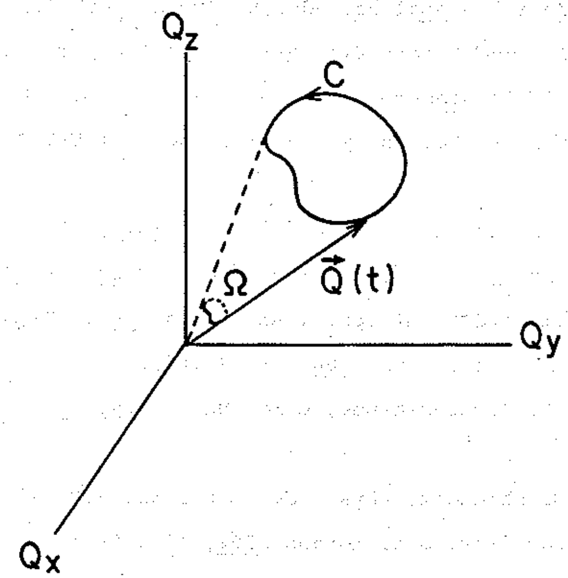

FIG. 1

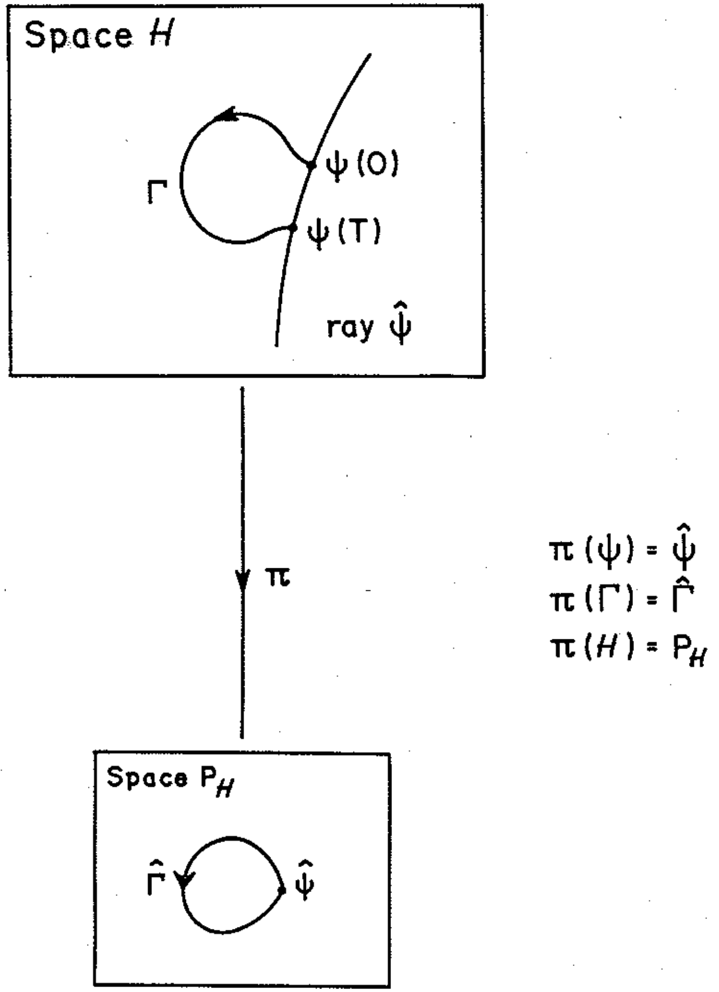
Space H
$\psi(0)$
Γ
$\psi(T)$
ray $\hat{\psi}$
π
$\pi(\psi) = \hat{\psi}$
$\pi(\Gamma) = \hat{\Gamma}$
$\pi(H) = P_H$
Space P_H
$\hat{\Gamma}$
$\hat{\psi}$

FIG. 2

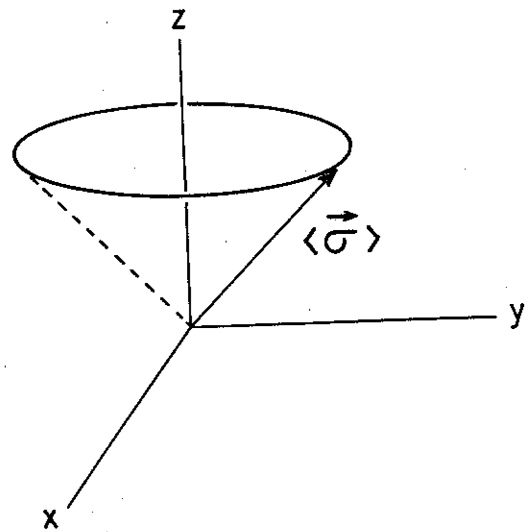
z
$\langle\vec{\sigma}\rangle$
y
x

FIG. 3