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ABSTRACT

A spinning particle in a gravitational field is shown to display a geometric phase. This provides a first-quantized example of a topological phase in a system with purely gravitational interactions. The connection with gravitational anomalies is brefly mentioned.

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INTRODUCTION

The number of systems displaying a quantum mechanical geometric phase [1] is by now quite large [2]. Initially defined within the adiabatic approximation, this concept was later generalized to cyclic [3] and non-linear evolutions [4]. In this note it is pointed out the existence of a new class of systems, namely those with purely gravitational interactions, which display geometric phases in certain circumstances. An analysis is made of the motion of a spinning particle in curved space and it is shown that a geometric phase arises in this case.

Geometric phases are known to spoil symmetries, making them anomalous [5]. For example, in the case of chiral symmetry the geometric phase is precisely the Wess-Zumino term [6]. Our result shows the emergence of these topological phases in a very simple context, and may help understand gravitational anomalies from a point of view somewhat different from the one explored so far [7].

After reviewing a definition of the geometric phase which is not restricted to adiabatic evolutions [3], we show that the equation of motion of a spinning particle in a gravitational field can be cast in a form which is known to lead to geometric phases.

2. GEOMETRIC PHASE FOR SPINNING PARTICLES

For the purposes of this paper, it is convenient to make use of a definition of the geometric phase due to Aharonov and

Anandam [3]. Its relation to Berry's phase is discussed in [3,8]. It generalizes Berry's phase to any cyclic evolution, i.e., one in which the wave function $\psi(\vec{x},t)$ satisfies

$$\psi(\vec{x},T) = e^{i\Lambda}\psi(\vec{x},0)$$

for a time T and real Λ.
The phase β is defined as that piece of the total phase Λ which is not dynamical, namely

$$\beta = \Lambda + \int dt < \psi(t) \mid H \mid \psi(t) >$$

where H is the Hamiltonian.

As defined, β is a property of the space P_H of rays $\hat{\psi}$ rather than the space H of states ψ. Indeed, each cyclic evolution defines a curves Γ in H that begins and ends on the same ray; it also defines a closed curve $\hat{\Gamma}$ in P_H, namely the projection of Γ. It is easy to show that β is a function of $\hat{\Gamma}$ only. Under adiabatic conditions for $H(t)$, β reduces to Berry's phase [3].

Now we show that the motion of a spinning particle in a gravitational field can be cast in a form which clearly gives rise to a geometric phase.

The spin vector S_μ of a particle in free fall obeys the equations [9]

$$\left(\frac{d}{d\tau} - \Gamma^\alpha_{\beta\nu}\frac{dx^\nu}{d\tau}g^{\mu\beta}\right)S_\mu = 0$$

and is further constrained by the condition $S^\alpha \dfrac{dx^\mu}{d\tau} = 0$.

For our purposes, it suffices to solve the equations within the Post-Newtonian approximation. The expansion of the metric in powers of the particle velocity, up to second order, leads to [9]

$$\frac{d\vec{\mathcal{S}}}{dt} = \vec{\Omega} \times \vec{\mathcal{S}}$$

where

$$\vec{\mathcal{S}} = (1+\phi)\vec{S} - \frac{1}{2}(\vec{v}\cdot\vec{S})\vec{S}$$

$$\vec{\Omega} = -\frac{1}{2}\vec{\nabla}\times\vec{\zeta} - \frac{3}{2}\vec{v}\times\vec{\nabla}\phi$$

and $\zeta_i = \mathcal{O}(v^3)$, $\phi = \mathcal{O}(v^2)$ are given by

$$g_{00} = -1 - 2\phi + \mathcal{O}(v^4)\,,\; g_{0i} = \zeta_i + \mathcal{O}(v^5)$$

In order to see that a geometric phase arises in this situation, it is simplest to consider first a spin $\frac{1}{2}$ particle at rest. If the gravitational field is stationary, i.e., the metric is time independent, but the interval is not invariant under time reversal - such as the one generated by a rotating body - then $\vec{\zeta}$ is not equal to zero and there is spin precession even for a particle at rest.

The Hamiltonian then becomes $-c\vec{\Omega}\cdot\vec{S}$, with $c = 1+\phi = \text{const}$, and the system is analogous to a spin precessing in a uniform magnetic field. The phase can be computed exactly as in [3]. Choosing the coordinate axes so that $\vec{\Omega}$ is along the z-axis, $H = -\mu\Omega\sigma_z$, with $\mu = \text{const}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If the initial state is $|\psi(0)> = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}$, then

$$|\psi(t)>= e^{i\mu\Omega\sigma_z t}\,|\psi(0)>= \begin{pmatrix} e^{i\mu\Omega t} & \cos(\theta/2) \\ e^{-i\mu\Omega t} & \sin(\theta/2) \end{pmatrix}$$

The spin vector $<\psi|\vec{\sigma}|\psi>$ precesses at an angle θ about the z-axis with a period $T = \pi/\mu\Omega$. Insertion of $<\psi(t)>$ in the expression for β gives

$$\beta(t) = (1-\cos\theta)\mu\Omega t$$

After one period, $\beta = (1-\cos\theta)\pi$, showing that the dynamical dependence drops out of $\beta(T)$. The phase is equal to half the solid angle subtended by the curve traced by the spin vector.

When the particle is moving, the Hamiltonian becomes $-\vec{\Omega}.\vec{S}$, where $\vec{\Omega}$ now depends on the position and velocity of the particle, and $\vec{S}$ is given by (3.2). As a consequence, the motion becomes quite complex, but the spin still precesses, and formula (2.3) remains valid. There is no a priori reason why β should vanish for an arbitrary circuit $\hat{\Gamma}$.

3. CONCLUSION

The result brings new members to the class of systems that display a geometric phase: those with purely gravitational

interactions. It is interesting also because it provides a first-quantized framework for the elusive issue of anomalies. Analogy with chiral symmetry would suggest that a quantum field theory of fermions in a gravitational field may develop anomalies. This turns out to be the case [7], but the analogy is not perfect; as pointed out in [10], the connection between geometric phases and anomalies in quantum mechanics holds only in the adiabatic regime. This connection is, however, different in field theory, in as much as it is independent of the adiabatic approximation.

Several aspects of this work can be pursued further: a more general treatment of the particle motion; in quantum field theory, the geometric phases should be related to indices of Dirac operators; an investigation of the effect of anomalous terms on the quantization of the gauge sector. We shall address these topics in a future publication.

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