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MHD EQUILIBRIUM EQUATION IN SYMMETRIC SYSTEMS

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ABSTRACT: In MHD symmectric systems the equilibrium physical quantities are dependent on two variables only. In these cases it is possible to find a magnetic surface function that has the same symmetry. Under the assumption that the metric determinant is also independent of a third, ignorable coordinate, a general MHD equilibrium equation in curvilinear coordinates is deduced. This equation is specially useful when non–orthogonal generalized coordinates are used.

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1 – INTRODUCTION

Ideal magnetohydrodinamics (MHD) is the most basic single–fluid model for determining the macroscopic equilibrium and stability properties of a plasma (Wesson 1978, Freidberg 1982). There is enough evidence that this model describes how magnetic, inertial and pressure forces interact within a perfectly conducting plasma (Bateman 1978, Goedbloed 1979).

The ideal MHD is used in magnetic fusion to describe static equilibria and to infer convenient magnetic geometries for confinement. In symmetric plasma systems, the field lines lie on a set of closed nested toroidal magnetic surfaces. These surfaces can be determined by solving the MHD equilibrium equation (Greene and Johnson 1965, Grad 1985).

Equilibrium equations for different symmetric plasma systems have appeared in the literature (Freidberg 1982). In particular, the well–known Grad–Shafranov (Hain et al 1957, Freidberg 1982) equation is written in terms of orthogonal coordinates and describes axisymmetric toroidal equilibrium.

In this article, a general MHD symmetric system is described by a magnetic surface function with the same symmetry as the considered equilibrium. This function is obtained from a general equilibrium equation in curvilinear coordinates.With this equation, once the curvilinear coordinates are chosen, the equilibrium equation in any geometry can be derived. This is specially useful when non–orthogonal generalized coordinates are used.

The general curvilinear coordinates used in this article are defined in section 2. A symmetric transversal magnetic flux Ψ and a function I that determines the transversal eletric current are introduced in sections 2 and 3. In section 4, a general equilibrium equation relating the equilibrium pressure to the surface functions Ψ and I is derived. Finally, systems well–known in the literature are considered as examples and the equilibrium equations in terms of conventional toroidal coordinates (Appendix A), helical coordinates (Appendix B) and natural coordinates (Appendix C) are derived from the general equation obtained in this article.

2. CURVILINEAR COORDINATES

In symmetric plasma confinement systems, all the equilibrium functions having a physical meaning are dependent on two variables only. Curvilinear coordinates are named u_1, u_2 and u_3. The surfaces $u_i = c_i$, where c_i is a constant, are coordinate surfaces. A coordinate curve u_k is a curve along which u_i and u_j ($i \neq j \neq k$) are constants.

The coordinates u_1 and u_2 are chosen in order to have the magnetic axis of the system coincident with a coordinate curve u_3 and u_2 is a transversal coordinate. u_3 will be an ignorable coordinate; longitudinal directions are given by coordinate curves u_3. In plasma confinement problems we have, usually, periodicity in u_2 and u_3. The following periodicity

$$u_3 = L(u_1, u_2) \tag{1}$$

is assumed.

An attempt is made to use the notations most familiar in the literature.

The covariant basis vectors are given by

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u_i} \tag{2}$$

where $\mathbf{e}_i$ is tangent to the u_i curve and the contravariant basis vectors are defined by

$$\mathbf{e}^i = \nabla u_i \tag{3}$$

where $\mathbf{e}^i$ is normal to the u_i surfaces. u_i, u_2 and u_3 are taken in order to satisfy

$$\mathbf{e}_i = \sqrt{g}\ \mathbf{e}^j \times \mathbf{e}^k \tag{4}$$

for any cyclic permutation (i, j, k), g is the determinant of the covariant metric tensor g_{ij}.

3. TRANSVERSAL MAGNETIC FLUX

Define $L\ \Psi(u_1, u_2)$ as the magnetic flux through a coordinate surface u_2 which extends from the magnetic axis to a coordinate curve u_3 and limited by $0 \leq u_3 \leq L$. On the magnetic axis, $u_1 = a$ and

$$\mathbf{B} = B^3\, \mathbf{e}_3\,. \tag{5}$$

Then:

$$L\ \Psi = \int_a^{u_1} du_1' \int_0^L \sqrt{g}\ B^2 du_3 \tag{6}$$

from which, it folows:

$$\frac{\partial \Psi}{\partial u_1} = \frac{1}{L} \int_0^L \sqrt{g}\, B^2\, du_3. \tag{7}$$

Taking account of the equation

$$\nabla \cdot \mathbf{B} = 0 \tag{8}$$

and assuming $B^1 = 0$ on the axis we can derive from expression (6):

$$\frac{\partial \Psi}{\partial u_2} = -\frac{1}{L}\int_0^L \sqrt{g}\, B^1\, du_3 \tag{9}$$

If $\sqrt{g}\, B^1$ and $\sqrt{g}\, B^2$ are independent of u_3, we find an expression for **B** in terms of Ψ:

$$\mathbf{B} = \frac{\mathbf{e}_3}{g_{33}} \times \nabla\Psi + B_3 \frac{\mathbf{e}_3}{g_{33}}. \tag{10}$$

Ψ = constant represents a magnetic surface because

$$\nabla\Psi \cdot \mathbf{B} = 0 \tag{11}$$

as can be seen using (10).

The magnetic flux can also be expressed in terms of the vector potential **A** using Stokes theorem:

$$\Psi = -\frac{1}{L}\int_0^L A_3 du_3 . \tag{12a}$$

A_3 is assumed to be zero on the axis. In symmetric systems:

$$\Psi\,(u_1, u_2) = -A_3(u_1, u_2). \tag{12b}$$

The only restriction to the gauge of **A** is in order to keep the same symmetry as the physical quantities.

4. CURRENT DENSITY

The current density satisfy the equations

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{and} \qquad \nabla \cdot \mathbf{J} = 0. \tag{13}$$

On the magnetic axis $J^1 = 0$. These equations are similar to the equations for **B**:

$$\nabla \times \mathbf{A} = \mathbf{B} \qquad \text{and} \qquad \nabla \cdot \mathbf{B} = 0 \tag{14}$$

and also $B^1 = 0$ on the axis.

Similar considerations must yield similar results. Let us define a function:

$$\mu_0 I = -\frac{1}{L}\int_0^L B_3\, du_3 \tag{15a}$$

which is an expression similar to (12a). In a symmetric case it would be:

$$\mu_0 I = -B_3(u_1, u_2). \tag{15b}$$

In this case, on the axis, B_3 is not zero.

The transversal current is given by:

$$L(I - I_{axis}) = \int \sqrt{g}\, J^2\, du_1\, du_3. \tag{16}$$

This expression can be compared to (6). Thus, considering the symmetry argument, the current density is expressed in terms of I as:

$$\mathbf{J} = \frac{\mathbf{e}_3}{g_{33}} \times \nabla I + J_3 \frac{\mathbf{e}_3}{g_{33}} \quad , \tag{17}$$

an expression similar to (10).

5. PRESSURE EQUILIBRIUM EQUATION

The MHD equilibrium theory, with scalar pressure P, considers the equations:

$$\nabla P = \mathbf{J} \times \mathbf{B} \tag{18}$$

and

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{19}$$

I, P and Ψ satisfy the relations:

$$\mathbf{B} \cdot \nabla \Psi = 0 \quad , \quad \mathbf{B} \cdot \nabla P = 0 \tag{20}$$

$$\mathbf{J} \cdot \nabla P = 0 \quad , \quad \mathbf{J} \cdot \nabla I = 0 \, , \tag{21}$$

what means that I, P and Ψ are surface quantities.

Using (18) together with (15b) and (17) a relation between these surface quantities is found:

$$\nabla P = -\frac{J_3}{g_{33}} \nabla \Psi + \frac{B_3}{g_{33}} \nabla I \tag{22}$$

J_3 can be taken from (19) and $B_3 = -\mu_0 I$. Substituting them in (22) and using (10) we find the final expression:

$$(\Delta^* \Psi) \nabla \Psi = -\mu_0 \, g_{33} \nabla P - \mu_0^2 \, I \, \nabla I + \mu_0 \, I \frac{g_{33}}{\sqrt{g}} \left[\frac{\partial}{\partial u_1} \left(\frac{g_{23}}{g_{33}} \right) - \frac{\partial}{\partial u_2} \left(\frac{g_{13}}{g_{33}} \right) \right] \nabla \Psi \tag{23}$$

where

$$\Delta^* \Psi \equiv \frac{g_{33}}{\sqrt{g}} \left\{ \frac{\partial}{\partial u_1} \frac{\sqrt{g}}{g_{33}} \left(g^{11} \frac{\partial \Psi}{\partial u_1} + g^{12} \frac{\partial \Psi}{\partial u_2} \right) + \frac{\partial}{\partial u_2} \frac{\sqrt{g}}{g_{33}} \left(g^{12} \frac{\partial \Psi}{\partial u_1} + g^{22} \frac{\partial \Psi}{\partial u_2} \right) \right\} \tag{24}$$

I and p are functions of Ψ only. Therefore, whenever $\nabla \Psi \neq 0$, the expression can be simplified to a scalar equation:

$$\Delta^* \Psi = -\mu_0 \, g_{33} \, P' - \mu_0^2 \, I \, I' + \mu_0 I \frac{g_{33}}{\sqrt{g}} \left[\frac{\partial}{\partial u_1} \left(\frac{g_{23}}{g_{33}} \right) - \frac{\partial}{\partial u_2} \left(\frac{g_{13}}{g_{33}} \right) \right] \tag{25}$$

which corresponds to the Grad–Shafranov equation (Freidberg 1982). Here the prime indicates differentiation with respect to Ψ. The quantity $\mu_0 g_{33} P + \mu_0^2 I^2/2$ must be continuous through a surface where $\nabla \Psi = 0$.

The components of the magnetic field are expressed in terms of Ψ as:

$$\sqrt{g} \, B^2 = \frac{\partial \Psi}{\partial u_1}$$

and (26)

$$\sqrt{g} \, B^1 = -\frac{\partial \Psi}{\partial u_2} \, .$$

The equation (24) can also be written as:

$$\Delta^* \Psi \equiv \nabla^2\Psi - \nabla\Psi \,.\frac{\nabla g_{33}}{g_{33}} \qquad (27a)$$

or

$$\Delta^* \Psi \equiv g_{33}\, \nabla.\, (\,\frac{\nabla\Psi}{g_{33}}\,). \qquad (27b)$$

The general equilibrium equation can be specially useful when non–orthogonal generalized coordinates are used.

In Appendix A toroidal pinches with axial symmetry are considered and the equilibrium equation in conventional toroidal coordinates (Shafranov 1960) is derived from equation (13). If the system presents a straight helical symmetry, the coordinates $u_1 = r$, $u_2 = \theta - \alpha z$ and $u_3 = z$ can be introduced, where α is the pitch of the helix. Using (23) the equilibrium equation as found in the literature (Freidberg 1982) is deduced straightforwardly. In these systems it is very likely to appear discountinuity surfaces where $\nabla\Psi = 0$ (See Appendix B). The well–known equilibrium equation using flux coordinates (Freidberg 1982) is obtained, also,very easily in Appendix C.

6. CONCLUSIONS

In this article a general MHD symmetric system has been considered and a magnetic surface function, with the same symmetry, has been introduced to describe it. An equilibrium equation satisfied by this function is deduced in curvilinear coordinates. This equation is a generalization of several particular MHD equilibrium equations valid for the equilibria considered in the literature. Therefore, given the curvilinear coordinates, the equilibrium equation for a magnetic surface function, in any geometry, can be derived. This

is specially useful if non–orthogonal coordinates are used. As examples, in the appendices, equilibrium equations well–known in the literature are obtained from the mentioned general equation presented in this article. The procedure followed in this paper resembles the one used to deduce an equilibrium equation for incompressible inotational steady fluid flow (Sparenberg 1989).

APPENDIX A

Toroidal pinch with axial symmetry

The equilibrium equation using conventional toroidal coordinate system (Shafranov 1960) (see Fig. A1).

$$u_1 = \xi \;, \qquad u_2 = \omega \;, \qquad u_3 = \varphi$$

is obtained in this appendix. The coordinates are defined by:

$$r = \frac{R_0' \;\sinh\xi}{\cosh\;\xi - \cos\omega} \qquad ; \qquad z = \frac{R_0' \;\sin\omega}{\cosh\;\xi - \cos\omega}$$

where r, z and φ are the polar cylindrical coordinates and R_0 is the major radius. If $\xi = \xi_0$ defines the toroidal surface, then $\cosh\,\xi_0 = R_0/b$, $R_0' = R_0\sqrt{1 - b^2/R_0^2}$.

The contravariant basis is:

$$e^1 = \nabla\xi = \frac{e_\xi}{h_\xi}$$

$$e^2 = \nabla\omega = \frac{e_\omega}{h_\omega}$$

$$e^3 = \nabla\varphi = \frac{e_\varphi}{h_\varphi}$$

with

$$h_\xi = h_\omega = \frac{R_0'}{\cosh_\xi - \cos\;\omega}$$

and

$$h_\varphi = h_\xi \sinh\,\xi.$$

e_ξ , e_ω and e_φ are unit vectors. The metric is given by:

$$\sqrt{g} = (e^1 \,.\, e^2 \times e^3)^{-1} = h_\xi\, h_\omega h_\varphi$$

$$g^{11} = \frac{1}{h_\xi^2} = g^{22}\;; \qquad g_{33} = h_\varphi^2$$

The equilibrium equation (25) becomes:

$$\frac{\sinh\xi}{h_\xi}\left(\frac{\partial}{\partial\xi}\frac{1}{h_\varphi}\frac{\partial\Psi}{\partial\xi} + \frac{\partial}{\partial\omega}\frac{1}{h_\varphi}\frac{\partial\Psi}{\partial\omega}\right) = -\mu_0 h_\varphi^2\, P' - \mu_0^2 I\, I' \tag{A–1}$$

If a function F is introduced as:

$$\Psi = [2(\cosh\,\xi - \cos\,\omega)]^{-1/2} F$$

we obtain the well known Grad–Shafranov equation (Freidberg 1982):

$$\frac{\partial^2 F}{\partial\xi^2} + \frac{\partial^2 F}{\partial\omega^2} - \coth\,\xi\,\frac{\partial F}{\partial\xi} + \frac{1}{4}F =$$

$$-\frac{2R_0'^2}{[2(\cosh\;\xi - \cos\;\omega)]^{3/2}}\left\{\frac{4R_0'^2\,\sinh^2\,\xi}{[2(\cosh\;\xi - \cos\;\omega)]^2}\,\mu_0\, P' + \mu_0^2\, I\, I'\right\}$$

The magnetic field is derived using (25):

$$h_\xi h_\varphi B_\omega = \frac{\partial \Psi}{\partial \xi}$$

and

$$h_\omega h_\varphi B_\xi = -\frac{\partial \Psi}{\partial \omega} \qquad \text{(A–2)}$$

APPENDIX B

Helical system with straight magnetic axis

The coordinates are:

$$u_1 = r \quad ; \quad u_2 = \theta - \alpha z \equiv u \quad ; \quad u_3 = z$$

where r, θ and z are the polar coordinates. Their Contravariant basis is:

$$\mathbf{e}^1 = \mathbf{e}_r\,, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_\theta}{r} - \alpha \mathbf{e}_z\,, \qquad \mathbf{e}^3 = \mathbf{e}_z$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are unit vectors in cylindrical coordinates. The covariant basis is:

$$\mathbf{e}_1 = \mathbf{e}_r\,, \qquad \mathbf{e}_2 = r\,\mathbf{e}_\theta \quad \text{and} \quad \mathbf{e}_3 = \mathbf{e}_z + \alpha r \mathbf{e}_\theta$$

The metric is given by:

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} + \alpha^2 & -\alpha \\ 0 & -\alpha & 1 \end{bmatrix} \quad ; \qquad g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & \alpha r^2 \\ 0 & \alpha r^2 & 1+\alpha^2 r^2 \end{bmatrix}$$

In this case the equilibrium equation (25) becomes (Freidberg 1982)

$$\frac{1 + \alpha^2 r^2}{r}\left(\frac{\partial}{\partial r}\left(\frac{r}{1+\alpha^2 r^2}\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \Psi}{\partial u^2}\right) = -(1 + \alpha^2 r^2)\mu_0\, P' - \mu_0^2 I\, I' +$$

$$+\mu_0 I \frac{1+\alpha^2 r^2}{r} \frac{\partial}{\partial r} \frac{\partial r^2}{1+\alpha^2 r^2} \tag{B–1}$$

and the magnetic field components are derived using the following equations:

$$-\frac{\partial\Psi}{\partial r} = B_\theta - \alpha r B_z \ , \qquad -\frac{\partial\Psi}{\partial u} = -r B_r \tag{B–2}$$

APPENDIX C

Natural coordinates

In natural system (Hamada 1962) the coordinate u_1 is a magnetic surface label; it is analogous to the minor radius of the torus. u_2 and u_3 are poloidal and toroidal cyclic coordinates, ranging from 0 to 2π.

The physical variables are:

magnetic pressure	$P(u_1)$
poloidal flux	$2\pi\,\chi\,(u_1) = \int_0^{u_1} du_1' \int_0^{2\pi} \sqrt{g}\; B^2\, du_3$
poloidal current	$2\pi\,(I - I_{axis}) = \int_0^{u_1} du_1' \int_0^{2\pi} \sqrt{g}\, J^2\, du_3$
toroidal flux	$2\pi\,\phi = \int_0^{u_1} du_1' \int_0^{2\pi} \sqrt{g}\, B^3\, du_2$
toroidal current	$2\pi\, J = \int_0^{u_1} du_1' \int_0^{2\pi} \sqrt{g}\; J^3\, du_2$
volume inside a toroidal magnetic surface	$(2\pi)^2 V(u_1) = \int_0^{u_1} du_1' \int_0^{2\pi} du_2 \int_0^{2\pi} \sqrt{g}\; du_3$

The equilibrium equation (22) becomes:

$$P' = -\frac{J_3}{g_{33}}\chi' - \frac{\mu_0 I}{g_{33}} I' \tag{C–1}$$

where the prime indicates derivation in respect to u_1. Multiplying equation (C–1) by $\sqrt{g}\, du_2\, du_3$ and integrating in a magnetic surface we get

$$P'V' = -\chi' J' - \phi' I' \tag{C–2}$$

The magnetic field and the current density are given by:

$$\mathbf{B} = \frac{\mathbf{e}_3}{g_{33}} \times \nabla\Psi - \frac{\mu_0 I}{g_{33}}\,\mathbf{e}_3 \qquad \text{(C–3)}$$

and

$$\mathbf{J} = \frac{\mathbf{e}_3}{g_{33}} \times \nabla I + \frac{J_3}{g_{33}}\,\mathbf{e}_3 \qquad \text{(C–4)}$$

In terms of the contravariant components (C–4) becomes:

$$\mathbf{J} = \frac{I'}{\sqrt{g}}\,\mathbf{e}_2 + J^3\mathbf{e}_3 \qquad \text{(C–5)}$$

Introducing the average value of $J^3\sqrt{g}$ in a magnetic surface:

$$<J^3\sqrt{g}> = \frac{1}{2\pi}\int_0^{2\pi} J^3\sqrt{g}\,du_2$$

and writing:

$$J^3\sqrt{g} = <J^3\sqrt{g}> + \overline{J^3\sqrt{g}}\ .$$

expression (C – 5) becomes:

$$\mathbf{J} = \frac{I'}{\sqrt{g}}\,\mathbf{e}_2 + \frac{J'}{\sqrt{g}}\,\mathbf{e}_3 + \frac{\overline{J^3\sqrt{g}}}{\sqrt{g}}\,\mathbf{e}_3$$

Defining a function ν by:

$$\frac{\partial\nu}{\partial u_2} = \overline{J^3\sqrt{g}}$$

the current density can be written:

$$\mathbf{J} = \nabla \times (-I\mathbf{e}^3 + J\,\mathbf{e}^2 - \nu\mathbf{e}^1)$$

and

B comes naturally as:

$$\frac{\mathbf{B}}{\mu_0} = -I\mathbf{e}^3 + J\mathbf{e}^2 - \nu\mathbf{e}^1 + \nabla\phi$$

where ϕ is the scalar potential in the absence of the plasma. This expression equated to (C–3) represents the equations commonly used to determine the flux coordinates (Hamada 1962, Hirshman 1982).

The only assumption taken is of symmetry of the system.

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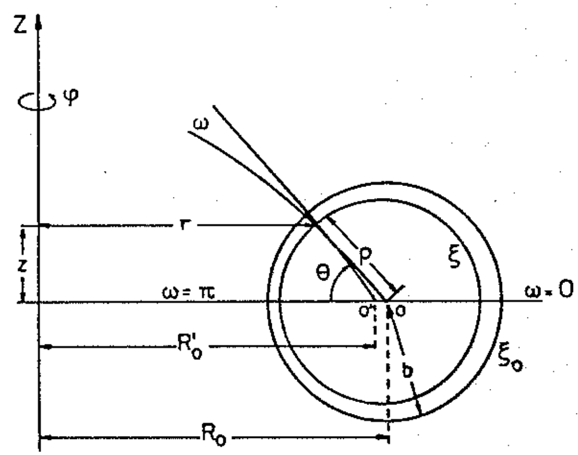

FIG. A1 – TOROIDAL COORDINATE SYSTEM.