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AVERAGE MAGNETIC SURFACES IN TOKAMAKS

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ABSTRACT

An average invariant which describes average magnetic surfaces for a system without symmetry was obtained. The system is a tokamak toroidal equilibrium perturbed by resonant helical windings. Analysis of the average surfaces showed that the magnetic islands move towards the plasma centre and decrease in width as the pressure increases.

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I – INTRODUCTION

When the magnetic field has symmetry, there are magnetic surfaces that can confine the plasma. Perturbing fields that break the symmetry may destroy these magnetic surfaces(1).

Resonant helical windings on a tokamak with the same helicity as the magnetic field are known to improve the equilibrium of the tokamak(2). However, these helical windings spoil the symmetry of the system.

The magnetic surfaces of a plasma are studied in a tokamak where the toroidal equilibrium is modified by resonant helical windings. As the amplitude of the current in the helical windings is much smaller than that of the plasma current, the helical windings are regarded as a perturbation of the equilibrium. The magnetic field is thus considered to be a superposition of the magnetic fields of the equilibrium and the helical windings.

The field lines can be described by means of a variational principle(3) using the vector potential of the system. Applying Noether's theorem to the Langrangian of the problem, one can conclude that, if all components of the vector potential are independent of one of the coordinates, the component of the vector potential corresponding to this coordinate is an invariant(3).

Our problem has no symmetry since the equilibrium has toroidal symmetry and the windings depend on an helical variable. Noether's theorem then cannot be applied directly because there is a dependence on all three coordinates. An average vector potential is therefore defined in order to allow an average invariant to be used to describe the problem approximately.

The method used to obtain the average surfaces is explained in Sect. II(4). Its application requires that the vector potentials of the equilibrium (Sect. III) and of the toroidal helical windings (Sect. IV) are calculated. In Sect. V the superposition of the helical windings on the equilibrium is analysed. The conclusions are given in Sect. VI.

II – AVERAGING METHOD

As the current in the helical windings is much smaller than the plasma current, the vector potential of the system $\mathscr{A}$ is the superposition of the vector potential of the equilibrium $\mathscr{A}^0$ and that of the perturbation $\vec{a}$:

$$\mathscr{A}(\rho, \theta, \varphi) \simeq \mathscr{A}^0(\rho, \theta) + \vec{a}(\rho, \theta\, \varphi), \tag{1}$$

where ρ, θ, φ are the local coordinates shown in Fig. 1.

The helical windings are described by the following equations:

$$u = m\theta - n\varphi = \text{constant} \tag{2a}$$

$$\rho = b. \tag{2b}$$

The average vector potential is defined as the average over the poloidal angle θ on a line where u is constant[4]:

$$\overline{A}_i(\rho, u) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, A_i\left[\rho, \theta, \varphi = \frac{m\theta - u}{n}\right] . \tag{3}$$

The average vector potential is of the form:

$$\overline{\mathscr{A}}(\rho, u) = \overline{A}_\rho(\rho, u)\, \hat{e}_\rho + \overline{A}_\theta(\rho, u)\, \hat{e}_\theta + \overline{A}_\varphi(\rho, u)\, \hat{e}_\varphi . \tag{4}$$

It is natural to use the variable u instead of φ. The average vector potential can then be written as[4]

$$\overline{\mathscr{A}}(\rho, u) = \overline{A}_\rho(\rho, u)\, \hat{e}_\rho + \left[\overline{A}_\theta(\rho, u) + \frac{m}{n}\overline{A}_\varphi(\rho, u)\right] \hat{e}_\theta - \frac{1}{n}\overline{A}_\varphi(\rho, u)\, \hat{e}_u . \tag{5}$$

In this coordinate system the average vector potential does not depend on the poloidal angle θ. Therefore, in accordance with Noether's theorem[4] the component θ of the average vector potential is an invariant:

$$\Psi(\rho, u) = \overline{A}_\theta(\rho, u) + \frac{m}{n}\overline{A}_\varphi(\rho, u) = \text{constant}. \tag{6}$$

Although Ψ is an exact invariant, it is not the exact invariant of the problem. The validity of the method derives from the fact that the average vector potential approximates the exact vector potential. The invariant of the approximate magnetic surface is thus an approximate invariant of the exact magnetic surface[4].

III – TOROIDAL EQUILIBRIUM

Shafranov analysed a plasma confined in a toroidal apparatus, using toroidal coordinates[5]. It is possible to transform these coordinates to local coordinates. It must be mentioned, however, that this transformation is not valid near the magnetic axis. The invariant which describes this equilibrium is[6]

$$\psi^0 = \frac{\mu_0 I_p R_0}{4\pi}\left[1 - \frac{\rho^2}{a^2}\right]\left[1 - \frac{\rho}{R_0}(\Lambda + 1)\cos\theta\right], \tag{7}$$

where the major and minor radii of the plasma are R_0 and a respectively (Fig. 1), I_p is the plasma current and Λ is defined by[7]

$$\Lambda = \beta_p + \frac{l_i}{2} - 1 . \tag{8}$$

In equation (8) β_p is the ratio between the kinetic pressure and the magnetic poloidal pressure of the plasma and l_i its internal inductance.

Figure 2 shows the magnetic surfaces described by curves of constant ψ^0, on the basis of the parameters of the TBR–1 tokamak[8]:

$$a = 8\ \text{cm},\ R_0 = 30\ \text{cm},\ I_p = 18\ \text{kA},\ \Lambda = 0.28$$

The function I related to the invariant ψ^0 is

$$I^2 = I_e^2 + 2\frac{R_0^2}{a^2} I_p^2\left[\frac{5}{4} - \Lambda\right]\left[1 - \frac{\rho^2}{a^2}\right]\left[1 - \frac{\rho}{R_0}(\Lambda + 1)\cos\theta\right], \tag{9}$$

where I_e is the current in the tokamak coils.

The magnetic field can be obtained with the expression

$$\mathscr{B} = \frac{1}{R}\left[\nabla\psi^0 \times \hat{e}_\varphi + \frac{\mu_0\ I(\psi^0)}{2\pi}\hat{e}_\varphi\right], \tag{10}$$

where R is given by (Fig. 1)

$$R = R_0 - \rho\cos\theta . \tag{11}$$

The vector potential can be calculated with the equation[10]

$$\mathscr{A} = \chi\,\nabla\theta - \Gamma\nabla\varphi , \tag{12}$$

where χ is the toroidal flux of the magnetic field (Fig. 3a),

$$\chi = \frac{1}{2\pi} \int \mathcal{B} \cdot d\vec{S}_T \ , \tag{13a}$$

and Γ the poloidal flux of the magnetic field (Fig. 3b),

$$\Gamma = \frac{1}{2\pi} \int \mathcal{B} \cdot d\vec{S}_p \ . \tag{13b}$$

Using expressions (12), (13), (10), (7) and (9) it is possible to calculate the vector potential of the equilibrium, making some approximations and keeping terms to the order of $(\rho/R_0)^2$. Averaging the expression obtained, employing definition (3), the average vector potential is finally found to be

$$\bar{A}^0_\rho = 0 \ , \tag{14a}$$

$$\bar{A}^0_\theta = \frac{\mu_0}{4\pi} \left\{ C \left[\frac{\rho}{2} + \frac{\rho^3}{8R_0^2} + \frac{\rho^5}{16R_0^4} \right] - \frac{D}{8C} \frac{\rho^3}{a^2} + \right.$$

$$\left. - \frac{D}{2C} \left[1 + (\Lambda + 1) \left[\frac{\rho^3}{8R_0^2} + \frac{\rho^5}{12a^2R_0^2} \right] \right] \right\} \ , \tag{14b}$$

$$\bar{A}^0_\varphi = \frac{\mu_0 I_p}{4\pi} \left\{ 1 + \frac{\rho^2}{2R_0^2} - \frac{\rho^2}{a^2} \left[1 + \frac{\rho^2}{R_0^2} \right] - \frac{\rho^2}{2R_0^2} (\Lambda + 1) - \frac{\rho^4}{2a^2R_0^2} (\Lambda + 1) \right\} , \tag{14c}$$

where

$$D = \frac{2I_p^2}{a^2} \left[\frac{5}{4} - \Lambda \right] \tag{15a}$$

and

$$C = \sqrt{D + \frac{I_e^2}{R_0^2}} \ . \tag{15b}$$

IV – TOROIDAL HELICAL WINDINGS

A number of equidistant thin conductors wound on a circular torus carrying currents I in alternating directions is considered. The torus has a minor radius b and the toroidal helical windings are characterized by the numbers of periods n and m of the helical field in the poloidal and toroidal directions, respectively.

The scalar potential for this system has already been obtained[11]:

$$\phi \simeq (-1)^{Nm+1} \left[1 + \frac{\rho}{2R_0} \cos\theta \right] \frac{m\mu_0 I}{\pi N} \left\{ \left[\frac{\rho}{b} \right]^{Nm} \text{senN}\,(m\theta - n\varphi) - \frac{b}{4R_0} \left\{ \left[\frac{\rho}{b} \right]^{Nm+1} \right. \right.$$

$$\left. \left. \frac{Nm+2}{Nm+1} \text{sen}\,[(Nm-1)\theta - Nn\varphi] + \left[\frac{\rho}{b} \right]^{Nm-1} \frac{Nm}{Nm+1} \text{sen}\,[(Nm+1)\theta - Nn\varphi] \right\} \right\} , \tag{16}$$

where N is the harmonic considered and m the number of current pairs. Each term of the scalar potential corresponds to a certain resonance. The first term relates to the resonance m/n and the other two to the secondary resonances (m–1)/n and (m+1)/n. Only the most important resonance is considered, since near the rational surfaces the contribution of the other rational surfaces is negligible. The expression taken for the scalar potential is then

$$\phi \simeq (-1)^{Nm+1} \left[1 + \frac{\rho \cos\theta}{2R_0} \right] \frac{m\mu_0 I}{\pi N} \left[\frac{\rho}{b} \right]^{Nm} \text{senN}\,(m\theta - n\varphi). \tag{17}$$

The magnetic field can be obtained by means of

$$\mathcal{B} = \nabla\phi \tag{18}$$

The same procedure adopted for the equilibrium is used to obtain the vector potential by means of the magnetic field fluxes. The vector potential is then calculated from eqs. (17), (18), (12) and (13). Using the definition of the average vector potential (3), one obtains

$$\bar{a}_\rho = 0 \tag{19a}$$

$$\bar{a}_\theta = (-1)^{Nm} \frac{\mu_0 I n}{2\pi^2 N R_0} \left[\frac{\rho}{b} \right]^{Nm} \frac{\rho}{Nm+2} \text{senN}\,(m\theta - n\varphi), \tag{19b}$$

$$\bar{a}_\varphi = (-1)^{Nm} \frac{\mu_0 I m}{\pi N} \left[\frac{\rho}{b} \right]^{Nm} \cos N\,(m\theta - n\varphi) \ . \tag{19c}$$

V – AVERAGE MAGNETIC SURFACES

As it is a superposition of the vector potential that is concerned here, the average vector potential is now known from eqs. (1), (14) and (19). An approximate invariant (6) can therefore be obtained by the method described in Sect. I[4]:

$$\psi = \frac{\mu_0}{4\pi}\left\{ C\left[\frac{\rho}{2}+\frac{\rho^3}{8R_0^2}+\frac{\rho^5}{16R_0^4}\right]-\frac{D\rho^3}{8Ca^2}-\frac{D}{2C}(\Lambda+1)\left[\frac{\rho^3}{8R_0^2}+\frac{\rho^5}{12a^2R_0^2}\right]\right.$$
$$\left.+\frac{m}{n}I_p\left[1+\frac{\rho^2}{2R_0^2}-\frac{\rho^2}{a^2}\left[1+\frac{\rho^2}{2R_0^2}\right]-\frac{\rho^2}{2R_0^2}(\Lambda+1)+\frac{\rho^4}{2a^2R_0^2}(\Lambda+1)\right]\right\}$$
$$+(-1)^{Nm}\left\{\frac{\mu_0 I n\rho}{2\pi^2 NR_0}\,\frac{1}{Nm+2}\left[\frac{\rho}{b}\right]^{Nm}\mathrm{sen}Nu+\frac{m^2\mu_0 I}{n\pi N}\left[\frac{\rho}{b}\right]^{Nm}\cos Nu\right\}, \qquad (20)$$

where u was defined in eq. (2a).

Figures (4a) and (4b) show curves of constant ψ using the parameters of TBR–1 and $m = 3$, $n = 1$, $N = 1$. In Fig. (4a) one uses $\Lambda = 0.28$, a typical value for TBR–1 tokamak[12]. In Fig. (4b) one employs $\Lambda = -1$, which corresponds to the limit of a zero ratio between the kinetic and magnetic pressures of the plasma.

From the figures it can be seen that in the limit of zero pressures, the average magnetic islands are bigger and are nearer the plasma boundary. The effect of the pressure on the island width can be verified by calculating the island width using usual expressions.

It should be stated that considering the pressure of the plasma is equivalent to a displacement of the magnetic axis, because when the pressure is not considered the magnetic axis coincides with the geometric axis. The effect of the island width has already been observed by considering the displacement of the magnetic axis analytically and numerically[13].

VI – CONCLUSIONS

The magnetic surfaces of a tokamak perturbed by resonant helical windings were analysed, although the system has no symmetry. An average invariant which approximately describes the problem was obtained. The average magnetic surfaces which form average magnetic islands were studied on the basis of the parameters of the TBR–1 tokamak.

It was found that the positions and widths of the islands depend on the plasma pressure. As the pressure increases the islands become smaller and shift to the centre of the plasma.

The same analysis was made using an expression for the helical windings without the toroidal effect[9]. It was concluded that the toroidal effect results in magnetic islands which are slightly smaller than that obtained from the cylindrical approximation[9].

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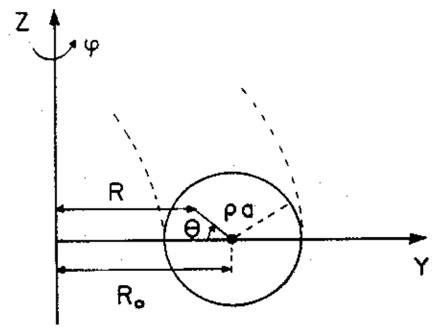

Fig. 1 – Coordinate system.

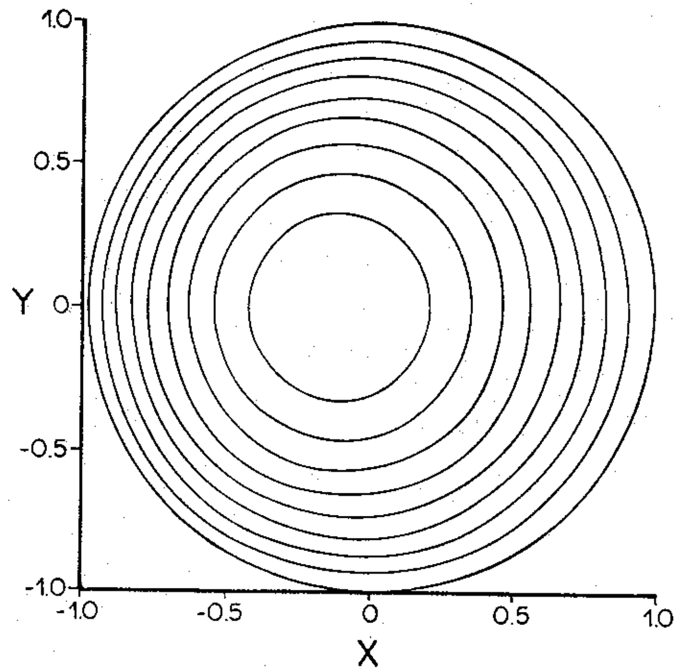

Fig. 2 – Surfaces of ψ^0 constant for $A = 0.28$, plasma current $I_p = 18$ kA, tokamak coil current $I_e = 600$ kA, major tokamak radius $R_0 = 30$ cm, minor tokamak radius $a = 8$ cm. The scales are normalized to the minor tokamak radius.

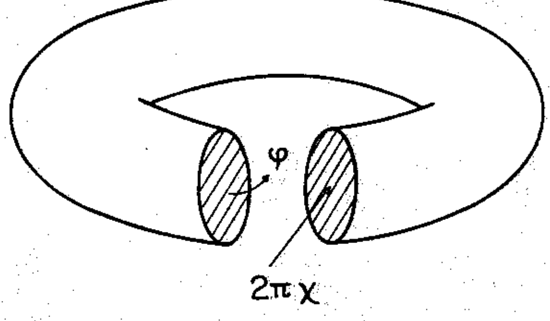

Fig. 3a – Toroidal flux in a tokamak.

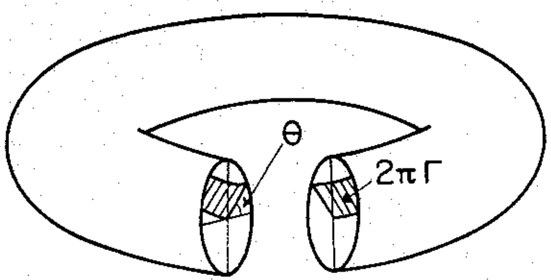

Fig. 3b – Poloidal flux in a tokamak.

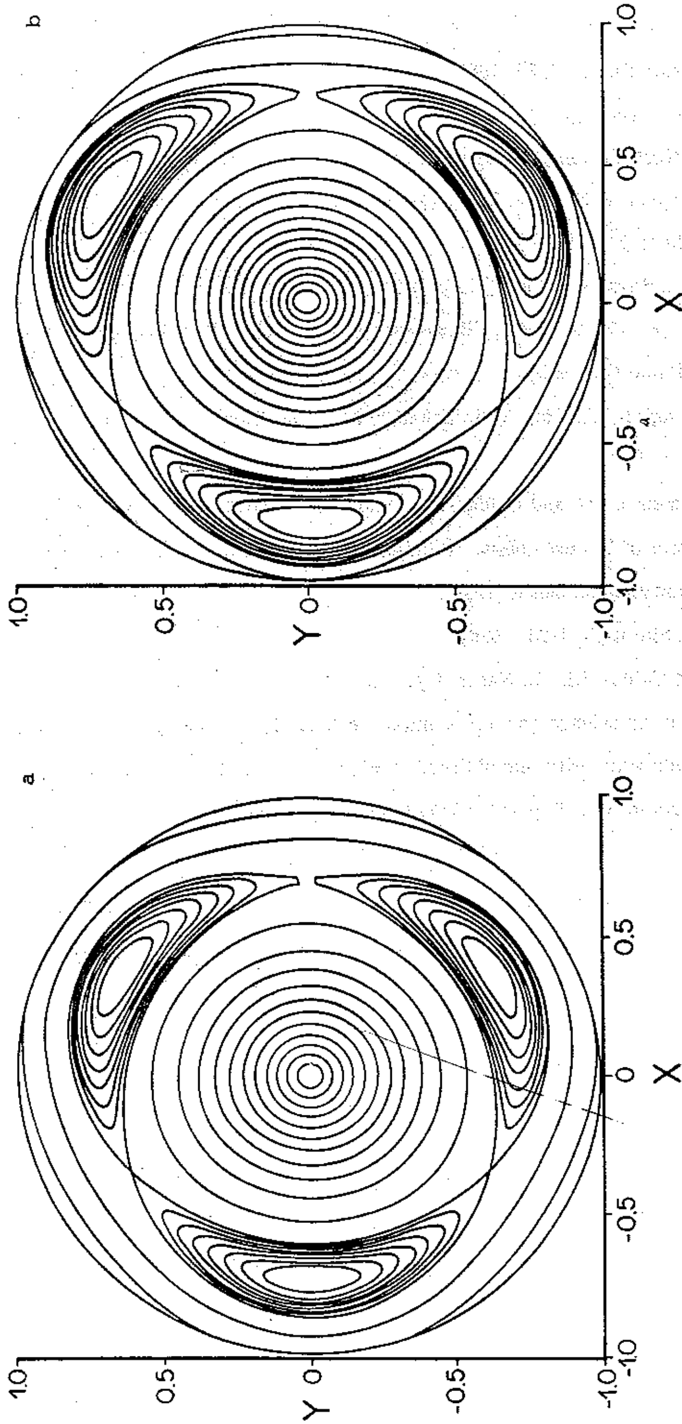

Fig. 4 – Surfaces of ψ constant, for $m = 3$, $n = 1$, $N = 1$, $I_p = 18$ kA, $I_e = 600$ kA, helical current $I = 100$ A, $a = 8$ cm, $R_o = 30$ cm, $\Lambda = 0.28$(a) and $\Lambda = -1.0$(b).