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QUANTUM MECHANICS OF RELATIVISTIC PARTICLES IN MULTIPLY CONNECTED SPACES AND THE AHARONOV-BOHM EFFECT

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Abstract

We consider the motion of free relativistic particles in multiply connected spaces. We show that if one of the spatial dimensions has the topology of a circle then the D dimensional spacetime is compactified to $D-1$ dimensions and the particle mass increases by an amount which is proportional to a quantum phase factor and inversely proportional to the radius of the circle. We also consider the relativistic Aharonov-Bohm effect and we show that the interference pattern is a universal characteristic due only to the topological properties of the experimental situation and not to the intrinsic properties of the particle. The propagators are calculated in both situations.

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1.- Introduction

Quantum mechanics in multiply connected spaces is an old subject in theoretical physics [1-4] and it is essential for a proper understanding of the Aharonov - Bohm effect [5], fractional statistics [6], topological field theories [7] and probably high T_c superconductivity [8].

In the path integral framework the multiconnected character of the manifold does appear when we consider all paths which belong to distinct homotopic classes. Then the propagation amplitude to go from the point X_1 to the point X_2 is given by

$$G[X_2, X_1] = \sum_n \Xi_n G_n[X_2, X_1], \tag{1}$$

where Ξ_n is some unitary representation of the n-th class of homotopy of the covering group of the manifold.

In the case of non-relativistic particles this problem has been studied by several authors [2,3,9,10,11] which have clarified the relation between the topological term $\alpha\dot{\theta}$ which we can add to the lagrangian and the multivalued character of the wave function.

These aspects, toghether with a rigorous mathematical treatment of quantum mechanics in multiply connected spaces [2] were essential for recents developments in fractional statistics and topological quantum field theories.

However, besides the large amount of work in this area, there is no discussion, to our knowledge, of the motion of relativistic particles in multipy connected spaces. Probably this gap is due to the fact that only recently a complete understanding of the quantization of the relativistic particle in terms of path integrals was reached. The crucial point is that the reparametrization invariance makes the path integral non trivial and the Faddeev - Popov [12] or Batalin - Fradkin - Vilkovisky [13] technique must necessarily be used.

Technically the problem of quantization of a generally covariant theory has to deal with the trouble of how to fix the gauge. The gauge fixing, as it was discussed some time ago by Teitelboim [14], must be consistent with the complete fixation of the gauge parameters at the initial and final points of the trajectory. Therefore, in this class of theories, we are lead naturally to non canonical gauge choices, like the proper time gauge.

The purpose of this paper is to discuss two examples involving free relativistic particles moving on spaces with non trivial topology using the path integral method. In particular, the analogous of the problem solved by Schulman [1,3] for a free non relativistic particle moving on a space with the topology of a circle S^1 will be solved explicitly in the proper time gauge. We will show that when we impose this topology the propagator of the relativistic particle is compactified from D to $D-1$ dimensions generating a mass proportional to the inverse radius of the circle and proportional to a quantum phase. This is done in section 2.

In section 3 we consider the relativistic Aharonov-Bohm effect. We will present a procedure which allow us to obtain an explicit expression for the propagator in momentum space. In the last part we compare our results with the non-relativistic ones showing that the interference pattern is the same in both cases. We then show that this is a general feature of the Aharonov-Bohm effect due to the topological properties of the experimental situation and independent of the intrinsic properties of the particle.

2.- The Free Relativistic Particle In a Manifold With The Topology S^1

Let us consider a relativistic particle with mass m in a D dimensional spacetime described by the following action

$$S=\int_{t_1}^{t_2}d\tau(P^{\mu}\dot{X}_{\mu}-N\mathcal{H}), \tag{2}$$

where N is the Lagrange multiplier and $\mathcal{H}$ is the first class constraint defined by

$$\mathcal{H}=\frac{1}{2}(P^2+m^2) \tag{3}$$

and P_{μ} is the momentum conjugated to X_{μ}.

The reparametrization invariance is given by the following transformations

$$\begin{aligned}\delta X^{\mu}&=\epsilon P^{\mu},\\ \delta P^{\mu}&=0,\\ \delta N&=\dot{\epsilon},\end{aligned} \tag{4}$$

which leave invariant the action (2) if at the end points we have

$$\epsilon(t_1)=\epsilon(t_2)=0. \tag{5}$$

As has been discussed in [14] a gauge choice compatible with (4) is the proper time gauge

$$\dot{N}=0. \tag{6}$$

To quantize the theory we will use the path integral formalism. The propagation amplitude for the particle to go from X_1 to X_2 is

$$G[X_2,X_1]=\int\mathcal{D}N\mathcal{D}X_{\mu}\mathcal{D}P^{\mu}\delta[\dot{N}]det(\partial_{\tau}^2)e^{i\int_{t_1}^{t_2}d\tau(P^{\mu}\dot{X}_{\mu}-N\mathcal{H})}. \tag{7}$$

The formal expression (7) must be evaluated specifying the boundary conditions, i.e., in this case the topology of S^1.

Let us assume that the topology of S^1 is imposed only in one of the coordinates, say X^1. Then we have the boundary conditions

$$\begin{aligned}X^0(t_1)&=X_1^0, & X^0(t_2)&=X_2^0,\\ X^1(t_1)&=X_1^1, & X^1(t_2)&=X_2^1+2\pi nl\\ X^i(t_1)&=X_1^i, & X^i(t_2)&=X_2^i,\end{aligned} \tag{8}$$

where $i=2,...,D-1$, l is the radius of the circle and n an integer.

The path integral (7) can now be evaluated explicitly. Notice that in (7) the factor $\delta[\dot{N}]$ states that only the zero mode ($N(0)$) of $N(t)$ contributes to the path integral and therefore the functional measure $\mathcal{D}N$ can de replaced by an ordinary measure $dN(0)$. The integration limits for $N(0)$, by causality requirements [14], is from 0 to ∞. The euclidianized determinant $\det(-\partial_{\tau}^2)$ can be evaluated by using the ζ-function regularization after we impose that the eigenfunctions of $-\partial_{\tau}^2$ vanishe at the end points of the trajectory*, the result being $\det(-\partial_{\tau}^2)=\Delta t=t_2-t_1$.

Then (7) becomes

$$G_n[X_2,X_1]=\int_0^{\infty}dT\int\mathcal{D}X_{\mu}\mathcal{D}P^{\mu}e^{i\int_{t_1}^{t_2}d\tau(P^{\mu}\dot{X}_{\mu}-N(0)\mathcal{H})}. \tag{9}$$

where $T=N(0)\Delta t$. In eq.(9) we added a lower indice n to G which means that $G_n[X_2,X_1]$ is the propagation amplitude for the n-th class of homotopy.

If we now write $P^{\mu}\dot{X}_{\mu}$ as $\frac{d}{dt}(P^{\mu}X_{\mu})-\dot{P}^{\mu}X_{\mu}$ and we integrate on X_{μ} we get a factor of $\delta[\dot{P}_{\mu}]$ which states that the momentum is conserved and allows the functional integral in P_{μ} to become an ordinary integral.

Using the boundary conditions (8) in order to evaluate the surface term, we get that (9) becomes

$$\begin{aligned}G_n[X_2,X_1]&=\int_0^{\infty}dT\int\frac{d^DP}{(2\pi)^D}e^{iP^{\mu}\Delta X_{\mu}+2in\pi lP_1-iT\mathcal{H}}\\ &=\int\frac{d^DP}{(2\pi)^D}\frac{e^{iP^{\mu}\Delta X_{\mu}+2inlP_1}}{P^2+m^2-i\varepsilon},\end{aligned} \tag{10}$$

where the factor $i\epsilon$ was introduced to guarantee the convergence of (9).

Using (1) we obtain the complete propagator as

$$G[X_2,X_1]=\sum_n\Xi_n\int\frac{d^DP}{(2\pi)^D}\frac{e^{iP^{\mu}\Delta X_{\mu}+2inlP_1}}{P^2+m^2-i\varepsilon}. \tag{11}$$

Following the arguments presented by Schulman [1] the Ξ_n can be easily obtained by observing that (11) depends only on the difference X_2-X_1, and so it is easy to show

* This has a natural explanation in the BFV formalism. In fact to impose that the eigenfunctions of $-\partial_{\tau}^2$ vanishe at the end points of the trajectory is equivalent to the imposition of homogeneous boundary conditions for the ghosts.

that $\Xi_n = e^{in\delta}$, where δ is a phase whose origin is strictly quantic and induced by the non-trivial topology of the coordinate X^1. Consequently (11) is

$$G[X_2, X_1] = \sum_{n=-\infty}^{\infty} \int \frac{d^D P}{(2\pi)^D} \frac{e^{iP^\mu \Delta X_\mu + in(2\pi l P_1 + \delta)}}{P^2 + m^2 - i\varepsilon},$$

$$= (2\pi) \int \frac{d^D P}{(2\pi)^D} \frac{e^{iP^\mu \Delta X_\mu}}{P^2 + m^2 - i\varepsilon} \delta[2\pi l P_1 + \delta],$$

$$= e^{\frac{-i\delta \Delta X_1}{2\pi l}} \int \frac{d^{D-1} P}{(2\pi)^{D-1}} \frac{e^{iP^\mu \Delta X_\mu}}{P^2 + \tilde{m}^2 - i\varepsilon}. \tag{12}$$

where now $\mu = 0, 2, \ldots, D-1$. Eq. (12) is the propagator for a particle of mass $\tilde{m}^2 = m^2 + \frac{\delta^2}{(2\pi l)^2}$ moving in a $D-1$ dimensional spacetime with trivial topology. Therefore, by imposing a non trivial topology in one of the spatial coordinates of the D dimensional spacetime we generate a compactification to $D-1$ dimensions. This has no analogie with the non relativistic case. It is also not analogous to the Cremmer - Scherk [15] compactification since our results are derived from a quantum phase δ and in this sense it is not equivalent to the compactification mechanism of the Kaluza - Klein theories [16] or strings theories [17]. It would be very interesting to extend these ideas to quantum field theories.**

3.- The Relativistic Aharonov - Bohm Effect

We now consider the Aharonov - Bohm effect. This is a two dimensional situation where the electron moves in a multiply connected space due to the presence of a cylinder of infinite lenght filled with a magnetic field. The relativistic effects should be observable by increasing the velocity of the electron. However, it seems that there is no experimental investigation in this direction [20].

Topogically speaking the situation described in fig.1 is not the most general possible since the electrons can follow other paths, e.g., going around the cylinder (see fig.2), and the problem is how to incorporate this into the path integral.

In principle it would be possible to extend the method developed by Inomota - Singh [21] and Gerry - Singh [22] to solve the non relativistic Aharonov - Bohm effect using the path integral. However, it seems more convenient to write directly the path integral for the relativistic problem and to factorize the corresponding integral in its space and time components (regarding the position vector X^μ).

** The literature about quantum field theories in multiply connected spaces is very restrict. Perhaps the first paper on this subject is due to Dowker [4]. More recently Hosotani [18] and Rajeev [19] have studied Yang - Mills theories in S^1.

The integral in X^0 can be solved easily and the spatial intagrals can be evaluated by using polar coordinates (see fig.2) as is done in [21] and [22]. Then, the formal path integral for the n-th class of homotopy in the proper time gauge has the following form

$$G_n[X_2, X_1] = \int \mathcal{D}N\mathcal{D}X^\mu det(N)^{-1} \delta[\dot{N}] det(N\partial_\tau{}^2) e^{i\int_{t_1}^{t_2} d\tau (\frac{1}{2N}\dot{X}^2 + \frac{m^2}{2}N)}. \tag{13}$$

The amplitude (13) is a lagrangian expression. The factor $det(N^{-1})$ has been incorporated to make the functional measure invariant under reparametrizations. It is also easy to see, by using the ζ-function regularization, that $det(N\partial_\tau{}^2)$ is factorizable and therefore the factors $det(N^{-1})$ and $\det(N)$ cancel among themselves. As before we have $det(-\partial_\tau{}^2) = \Delta t$.

The factor $\delta[\dot{N}]$ allow us to replace the path integral in N by an ordinary integral in $N(0)$ with integration limits from 0 to ∞. Then (13) becomes

$$G_n[X_2, X_1] = \int_0^\infty dT e^{i\frac{m^2 T}{2}} \int \mathcal{D}X^0 e^{i\int_{t_1}^{t_2} d\tau \frac{-\dot{X}_0^2}{2N(0)}} \int \mathcal{D}^2\mathbf{X} e^{i\int_{t_1}^{t_2} d\tau \frac{\dot{\mathbf{X}}^2}{2N(0)}}. \tag{14}$$

To solve the integrals in X^0 and $\mathbf{X}$ we consider that they are formally equal to the integrals for the non relativistic particle in one and two dimensions with mass $m = N(0)^{-1}$. So the X^0 integral is simply

$$\left(\frac{1}{2\pi T}\right)^{\frac{1}{2}} e^{\frac{-i(\Delta X_0)^2}{T}}. \tag{15}$$

The $\mathbf{X}$ integral is more complicated because we have to take into account that the particle can go around the cylinder. This can be done and we refer to the papers [21] and [22] for details (see also [23]). The result for the integral (14) is

$$G_n[X_2, X_1] = \int_0^\infty dT \int_{-\infty}^{+\infty} d\lambda \frac{1}{(2\pi T)^{\frac{3}{2}}} e^{\frac{im^2 T}{2} - i\frac{(\Delta X_0)^2}{2T} + \frac{i}{2}\frac{(R^2 + R'^2)}{T} + i\lambda(\phi + 2\pi n)} I_{|\lambda|}(i\frac{RR'}{T})$$

where ϕ is the angle difference between R and R' (see fig.2) and $I_{|\lambda|}$ is the modified Bessel function.

Following the same arguments used in the the former section to determinate Ξ_n in eq.(1) we obtain the complete propagator

$$G[X_2, X_1] = \int_0^\infty dT \frac{1}{(2\pi T)^{\frac{3}{2}}} e^{\frac{im^2 T}{2} - i\frac{(\Delta X_0)^2}{2T} + \frac{i}{2}\frac{(R^2 + R'^2)}{T}} \times$$

$$\times \sum_{n=-\infty}^{\infty} (-i)^{|n+\alpha|} e^{-i(n+\alpha)\phi} J_{|n+\alpha|}(\frac{RR'}{T}) \tag{16}$$

where in this case the phase is $\alpha = e\Phi$, where Φ is the magnetic flux and e is the electric charge. In eq. (16) we made use of the identity

$$I_\nu(ix) = (i)^\nu J_\nu(x)$$

Eq.(16) can also be writen explicitly in momentum space

$$G[X_2, X_1] = \int d^3P \int_0^\infty dT e^{iP^\mu \Delta X_\mu - i\frac{T}{2}(P^2+m^2)} \sum_{n=-\infty}^{\infty} (-i)^{|n+\alpha|} e^{-i(n+\alpha)\phi} J_{|n+\alpha|}(\frac{RR'}{T}). \tag{17}$$

Making the change of variables $T = \frac{1}{z}$, using the identity

$$J_{\nu-1}(x) + J_{\nu+1}(x) = 2\nu\, x J_\nu(x)$$

(ν arbitrary) and using the integral [24]

$$\int_0^\infty dz z^{-1} e^{\frac{\alpha}{z}} J_\nu(\beta z) = 2J_\nu(\sqrt{\alpha\beta})K_\nu(\sqrt{\alpha\beta}), \quad Re(\alpha) > 0, \quad \beta > 0$$

it follows that eq.(16) has the following form

$$G[X_2, X_1] = \lim_{\epsilon \to 0} \sum_{n=-\infty}^{\infty} \frac{(-i)^{|n+\alpha|}}{|n+\alpha|} e^{-i(n+\alpha)\phi} \int \frac{d^3P}{(2\pi)^3} e^{iP^\mu \Delta X_\mu} \times$$

$$\times (J_{|n+\alpha|-1}(\sqrt{RR'\rho}) + J_{|n+\alpha|+1}(\sqrt{RR'\rho}))(K_{|n+\alpha|-1}(\sqrt{RR'\rho}) + K_{|n+\alpha|+1}(\sqrt{RR'\rho})) \tag{18}$$

with $\rho = \epsilon - i(P^2 + m^2)$.

4.- Conclusions

We have formulated the path integral quantization of a relativistic particle on a spacetime with non trivial topology and writen explicitly expressions for the propagators. For the free relativistic particle we have found a quantum mechanism of mass generation which would be very interesting to see it extended to quantum field theories.

We also discussed the relativistic Aharonov - Bohm effect and calculated the propagator. We remark that its form is the same as the non relativistic one. If we write

$$G[X_2, X_1] = \sum_{n=-\infty}^{\infty} (-i)^{|n+\alpha|} e^{-i(n+\alpha)\phi} F_{|n+\alpha|} \tag{19}$$

we recover the relativistic propagator (17) for

$$F_{|n+\alpha|} = \int d^3P \int_0^\infty dT e^{iP^\mu \Delta X_\mu - i\frac{T}{2}(P^2+m^2)} J_{|n+\alpha|}(\frac{RR'}{T}) \tag{20}$$

while for the non relativistic case [23],

$$F_{|n+\alpha|} = \frac{m}{2\pi i} e^{\frac{2mi}{\tau}(R^2+R'^2)} J_{|n+\alpha|}(\frac{mRR'}{\tau}) \tag{21}$$

where τ is the time that the particle takes to go from X_1 to X_2.

This fact shows that we have a universal form for the propagator of the Aharonov - Bohm effect. If we had considered the relativistic spinning particle, for example, only the factor $F_{|n+\alpha|}$ in (19) would change. If we now apply the Poisson summation formula to (19) and make a change of variables [25] we can rewrite (19) as

$$G[X_2, X_1] = \sum_{m=-\infty}^{\infty} T_m e^{-2\pi i m\alpha} \tag{22}$$

where

$$T_m = \int_{-\infty}^{+\infty} d\lambda (-i)^{|\lambda|} e^{-i\lambda\phi} F_{|\lambda|} \tag{23}$$

so that all dependence on the flux Φ is in the exponential factor in (22). This means then that the interference pattern will be the same whichever particle we take, relativistic, spinning particle, etc. So the interference pattern of the Aharonov - Bohm effect is universal.

Of course, this requires a note of caution since we considered an ideal impenetrable cylinder and experimentaly an electron has a finite probability of entering the cylinder. In this case relativistic effects and spin couplings can change the interference pattern [26]. We also assumed that the cylinder is infinite in lenght but it is know that a finite cylinder changes the usual results [27].

REFERENCES

[1] L.S.Schulman, Phys.Rev. **176** 1558 (1968)
[2] M.G.G.Laidlaw and C.M.deWitt, Phys.Rev. **D3** 1375 (1971)
[3] L.S.Schulman, **Techniques and Applications of Path Integration** (Wiley, 1981)
[4] J.S.Dowker, J.Phys. **A5** 304 (1972)
[5] Y.Aharonov and D.Bohm, Phys.Rev. **115** 485 (1959)
[6] F.Wilczek, Princeton preprint (1990)
[7] E.Witten, Comm.Math.Phys **117** 353 (1988)
[8] Y-H.Chen, F.Wilczek, E.Witten and B.I.Halperin, Int.J.Mod.Phys. **B3** 1001 (1989)
[9] H.J.Rothe, Nuovo Cim. **62A** 54 (1981)
[10] H.J.Rothe and J.A.Swieca, Nucl.Phys. **B149** 237 (1979)
[11] B.H.Lee, S.H.Lee, E.J.Weinberg and K.Lee, Phys.Rev.Lett. **60** 2231 (1988)
[12] e.g. L.D.Faddev and A.A.Slavnov, **Introduction to Quantum Theory: Gauge Fields** (Benjamin, 1980)

[13] For a review see: M.Henneaux, Phys.Rep. **126** 1 (1985)
[14] C.Teitelboim, Phys.Rev. **D25** 3152 (1982)
[15] E.Cremmer and J.Scherk, Nucl.Phys. **B110** 61 (1977)
[16] For a review see M.Duff, B.E.W.Nilsson and C.N.Pope, Phys.Rep. **130** 1 (1986)
[17] M.B.Green, J.H.Schwarz and E.Witten, **Superstring Theory** (Cambridge, 1987)
[18] Y.Hosotani, Ann.Phys.(NY) **190** 233 (1989)
J.E.Hetrick and Y.Hosotani, Phys.Lett. **230B** 88 (1989)
[19] S.G.Rajeev, Preprint UR-1069 (1989)
[20] S.Olariu and I.Popescu, Rev.Mod.Phys. **57** 339 (1985)
[21] A.Inomata and V.A.Singh, J.Math.Phys. **19** 2318 (1978)
[22] C.Gerry and V.A.Singh, Phys.Rev. **D20** 2550 (1979)
[23] A.Y.Shiekh, Ann.Phys.(NY) **166** 299 (1986)
[24] I.S.Gradshtein and I.M.Ryzhik, **Table of Integrals, Series and Products** (Academic Press, 1980)
[25] M.V.Berry, Eur.J.Phys. **1** 1240 (1980)
[26] C.R.Hagen, Preprint UR-1145 (1990) and references therein
[27] G.N.Afanesiev and V.M.Shilov, J.Phys. **A22** 5195 (1989)

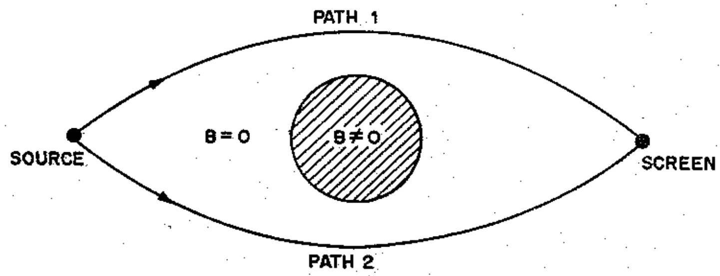

Fig. 1

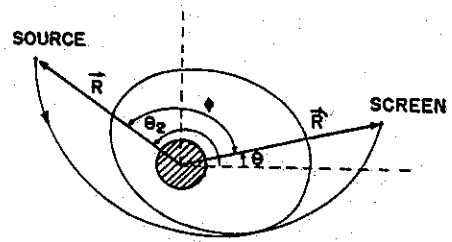

Fig. 2