SYMMETRIES OF CHERN-SIMONS THEORY IN LANDAU GAUGE

P.H. Damgaard
The Niels Bohr Institute, University of Copenhagen
Blegdamsvej 17, DK-2100 Copenhagen, Denmark

V.O. Rivelles
Instituto de Física, Universidade de São Paulo

Abril/1990
Symmetries of Chern-Simons Theory in
Landau Gauge

P.H. DAMGAARD
The Niels Bohr Institute
University of Copenhagen
Blegdamsvej 17
DK-2100 Copenhagen
Denmark

V.O. RIVELLES*
Instituto de Física
Universidade de São Paulo
C.Postal 20516
01498 São Paulo
Brazil

Abstract

In $d = 3$ dimensions the $ISO(d|2)$ algebra admits a non-trivial modification. We show that 3-dimensional Chern-Simons theory in Landau gauge has a global symmetry based on this large Lie superalgebra.

Interest in the quantization of a gauge theory based on a pure Chern-Simons action in $d = 3$ dimensions arises for several reasons. This type of theory is one example of a quantum field theory which is *topological* in the sense of being independent of the space-time metric on the three-manifold on which it is defined. As such it certainly deserves to be studied in its own right. Another approach to Chern-Simons theory comes from its connection to conformal field theories in one dimension lower[1,2]. This latter connection, and its relation to knot theory, has recently been explored from several points of view, and by now various groups have also performed explicit one and two loop perturbative computations in Chern-Simons theory, some of them motivated by the relation of this gauge theory to 2-dimensional conformal field theories[3-7].

One remarkable property of Chern-Simons theory is the existence of a so far unexplained new "supersymmetry" when this theory is considered in Landau gauge[3,7,9]. It is only a symmetry once the gauge has been fixed (it couples non-trivially between the Chern-Simons term and the ghost sector), and it appears difficult to extend it to other gauges. This is a quite puzzling situation which ought to be better understood.

In this note we shall demonstrate that this new symmetry in fact is only one part of a larger invariance group, based on a slightly modified $ISO(d|2)$ algebra. Before doing this, let us recapitulate a few facts about the ordinary $ISO(d|2)$ algebra, and show why the case $d = 3$ is rather special. Many details about the $ISO(d|2)$ algebra, and in particular its importance in string theory, can be found in refs.[10,11].

The usual inhomogeneous orthosymplectic $ISO(d|2)$ algebra has bosonic generators $J_{\mu\nu} = -J_{\nu\mu}, \mu = 0, 1, \ldots, d - 1; J_{AB}, A, B = 1, 2$ (or $+ \cdot$) and $P_{\mu}$. The fermionic generators are $J_{\alpha\delta}$ and $P_{\alpha}$. It is a superspace generalization of the usual Poincaré algebra, with $J_{\mu\nu}$ generating ordinary Lorentz transformations, and $P_{\mu}$ generating the usual space-time translations. The objects $J_{AB}$ and $P^A$ are the corresponding generators in the anticommuting coordinates. The full algebra is defined by the following non-trivial (anti)commutation relations, with $\epsilon_{\mu\nu\rho}$ denoting the space-time metric in the d-dimensional space, and $C_{AB}$ being the symplectic metric in the two extra dimensions:

\begin{align*}
[J_{\mu\nu}, J_{\alpha\delta}] &= \epsilon_{\mu\alpha}J_{\nu\delta} - \epsilon_{\mu\delta}J_{\nu\alpha} + \epsilon_{\nu\alpha}J_{\mu\delta} - \epsilon_{\nu\delta}J_{\mu\alpha} \\
[J_{AB}, J_{CD}] &= C_{AC}J_{BD} + C_{AD}J_{BC} + C_{BC}J_{AD} + C_{BD}J_{AC} \\
[J_{\mu\alpha}, J_{\nu\delta}] &= -\epsilon_{\mu\nu}J_{\alpha\delta} + \epsilon_{\mu\delta}J_{\nu\alpha} \\
[J_{AB}, J_{\alpha\delta}] &= C_{AC}J_{BD} + C_{BC}J_{AD} \\
[P_{\mu}, J_{\alpha\delta}] &= \frac{1}{2} [P_{\mu}, J_{\alpha\delta}] = 0 \\
\end{align*}

(1) (2) (3) (4) (5)

\begin{align*}
[P_{\mu}, P_{\nu}] &= \epsilon_{\mu\nu}P_{\alpha} - \epsilon_{\mu\alpha}P_{\nu} \\
[J_{\mu\nu}, P_{\alpha}] &= \frac{1}{2} [J_{\mu\nu}, P_{\alpha}] = 0 \\
[J_{AB}, P_{\delta}] &= C_{AC}P_{D} + C_{BC}P_{A} \\
[J_{\alpha\delta}, P_{\mu}] &= \epsilon_{\mu\alpha}P_{\delta} \\
[J_{\mu\alpha}, P_{\delta}] &= -C_{AB}P_{\delta} \\
\end{align*}

(6) (7) (8) (9) (10) (11)

with all other (anti)commutation relations vanishing. The special situation arising in $d = 3$ dimensions is due to the possibility of also using the $\epsilon_{\mu\nu\rho}$ symbol in the defining equations of the algebra. One point in the above sequence of equations where this is
possible is the case of \( \{ J_{\mu A}, J_{\nu B} \} \); consider the replacement of eq.(3) by

\[
\{ J_{\mu A}, J_{\nu B} \} = \epsilon_{\mu \nu \rho} C_{AB} P^\rho
\]  
(12)

Making the double replacement of \( \mu \leftrightarrow \nu \) and \( A \leftrightarrow B \), we see that this could be a consistent ansatz. But of course we also need to check if all Jacobi identities can be satisfied as well. This is found to be the case provided we simultaneously change eq.(10) to

\[
[J_{\mu A}, P_{\mu} ] = 0
\]  
(13)

So the system of equations (1)-(11), with the modifications (12) and (13), does indeed define a non-trivial modification of the \( ISO(3) \otimes Sp(2) \) algebra, particular to \( d = 3 \) dimensions. Notice that this is an extension of the usual space-time supersymmetry algebra, where the anticommutators of the supersymmetry generators yields a space-time translation. Here we have an extended supersymmetry generator which carries an additional space-time index.

To specify the modified algebra more precisely, we introduce an ordinary Lie algebra \( G_0 \) defined by the semisimple product between \( SO(3) \otimes Sp(2) \) and space-time translations. This is just \( ISO(3) \otimes Sp(2) \). Our graded Lie algebra \( G \) is then given by \( G = G_0 \oplus G_1 \), where the \( G_0 \)-module \( G_1 \) is defined by \( G_1 = \{ P_{\mu}, J_{\mu A} \} \). It is easy to see that with a product \( \star \) defined by the commutation relations (1)-(11) (and the modifications (12) and (13)), we have \( G_1 \star G_1 \subseteq G_1 \forall \mu, \nu \in \mathbb{Z}(2) \), and all Jacobi identities satisfied, \( G \) does indeed form a Lie superalgebra. There is no change of basis (redefinition of the generators) which can relate this super Lie algebra to that of \( ISO(3) \otimes Sp(2) \), although the two algebras are so similar in form. (This would in any case be rather surprising, since our algebra is very specifically related to \( d = 3 \) dimensions, while that of \( ISO(3) \otimes Sp(2) \) depends on dimensionality in a completely trivial way).

Curiously, our super Lie algebra can also be obtained by an Inönü-Wigner contraction of the simple exceptional superalgebra \( D(2|1; \alpha) \) (in Kac's classification scheme[12]). \( D(2|1; \alpha) \) has bosonic generators \( Q_{a}^\mu (\mu = 1, 2, 3 ; a = 1, 2, 3) \) and fermionic generators \( R_{ABC}(A, B, C, \bar{C}, \bar{A}, \bar{B}, \bar{C}, \bar{A}, \bar{B}, \bar{C}) \). The commutation relations can be written as

\[
[Q_a^\mu, Q_b^{\nu}] = \delta^\nu{}_{\nu'} \epsilon_{\mu \nu \rho} Q_a^{\rho}
\]

\[
[Q_a^\mu, R_{ABC}] = \frac{1}{2} \epsilon_{abc} \sigma^{\mu} Q_a^\nu
\]

\[
[Q_b^{\nu}, R_{ABC}] = \frac{1}{2} \epsilon_{abc} \sigma^{\nu} Q_a^\mu
\]

\[
[Q_a^\mu, R_{ABC}] = \frac{1}{2} \epsilon_{abc} \sigma^{\mu} Q_a^{\nu}
\]

\[
[R_{ABC}, R_{DEF}] = i \alpha_1 (C_{a}^{\nu} C_{B}^{\nu} C_{C}^{\nu} Q_a^\mu + i \alpha_2 C_{A}^{\nu} C_{B}^{\nu} C_{C}^{\nu} Q_a^\mu + i \alpha_3 C_{A}^{\nu} C_{B}^{\nu} C_{C}^{\nu} Q_a^\mu)
\]

(14)

where \( \sigma^a \) are the Pauli matrices and \( C \equiv \sigma^2 \). The real numbers \( \alpha_1 \) are arbitrary, except for the constraint \( \alpha_1 + \alpha_2 + \alpha_3 = 0 \).

We can realize this superalgebra if we modify our algebra as follows (let \( J_{\mu A} = \epsilon_{\mu \nu} P_{\nu} \) in \( d = 3 \) dimensions):

\[
[J_{\mu A}, J_{\nu B}] = \epsilon_{\mu \nu \rho} C_{AB} P_{\rho}
\]

\[
[P_{\mu}, P_{\nu}] = \epsilon_{\mu \nu \rho} P_{\rho}
\]

\[
[J_{\mu A}, P_{\nu}] = m^2 \epsilon_{\mu \nu \rho} J_{\rho}
\]

\[
[J_{\mu A}, J_{\nu B}] = \epsilon_{\mu \nu \rho} J_{\rho}
\]

\[
[J_{\mu A}, J_{\nu B} ] = 0
\]

\[
[P_{\mu}, P_{\nu}] = \frac{m}{\alpha_1 - \alpha_2} \alpha_0 \eta_{\mu \nu} P_{A}
\]

\[
[P_{\mu}, J_{\nu A}] = -\frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} \eta_{\mu \nu} P_{A}
\]

\[
[P_{\mu}, J_{\nu A}] = m \frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} J_{\nu A}
\]

\[
[J_{\mu A}, J_{\nu B}] = \epsilon_{\mu \nu \rho} C_{AB} \left( P_{\rho} + m \frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} J_{\rho} \right)
\]

\[
[J_{\mu A}, P_{\nu}] = -C_{AB} \left( P_{\nu} + \alpha_1 - \alpha_2 J_{\nu A} \right)
\]

\[
[J_{\mu A}, J_{\nu B}] = C_{AB} J_{\mu A} + C_{AD} J_{\nu C} + C_{BD} J_{\nu A} + C_{CD} J_{\mu A}
\]

\[
[J_{\mu A}, P_{\nu}] = 0
\]

\[
[J_{\mu A}, J_{\nu B}] = C_{AD} J_{\nu B} + C_{BD} J_{\nu A}
\]

(15)

When \( m \to 0 \) we get back our superalgebra. The transformation between our generators and those of \( D(2|1; \alpha) \) listed above is

\[
Q_a^\mu = \frac{1}{2} J_{a} - \frac{P_{a}}{m}
\]

\[
Q_a^\nu = \frac{1}{2} J_{a} - \frac{P_{a}}{m}
\]

\[
Q_a^\mu = \frac{1}{2} (C_{a}^{\nu} C_{B}^{\nu} C_{C}^{\nu} Q_a^\mu + i \alpha_2 C_{A}^{\nu} C_{B}^{\nu} C_{C}^{\nu} Q_a^\mu + i \alpha_3 C_{A}^{\nu} C_{B}^{\nu} C_{C}^{\nu} Q_a^\mu)
\]

(16)

This contraction transformation is indeed singular when \( m \to 0 \). However, as we have seen, the algebra itself is well defined in this limit, where it reduces to ours. What is the relation between this large graded Lie algebra and 3-dimensional Chern-Simons theory? Let us first fix notation. We define

\[
S_{CS}[A_{\mu}] = \frac{k}{4 \pi} \int \mathrm{d}^3 x \mathrm{Tr} (\epsilon_{\mu \nu \rho} A_{\mu} \partial_{\nu} A_{\rho} + \frac{2}{3} A_{\mu} A_{\nu} A_{\rho})
\]

where \( k \) is restricted to be an integer (since otherwise the exponential of the action is not invariant under large gauge transformations). As in the perturbative treatment[8], it is convenient to introduce a more conventional coupling constant \( g^2 \equiv 4 \pi k^2 \) and then absorb it partially by a rescaling of the gauge potential, \( A_{\mu} \to g A_{\mu} \). After gauge fixing into Landau gauge \( (\partial_\mu A_\mu = 0) \) the resulting action is of the form \( S = S_{CS}[A_{\mu}] + S_{F}(A_{\mu}, \xi, \phi) \):

\[
S = \int \mathrm{d}^3 x \mathrm{Tr} (\epsilon_{\mu \nu \rho} (A_{\mu} \partial_{\nu} A_{\rho} + \frac{2}{3} A_{\mu} A_{\nu} A_{\rho}) + \xi \partial_\mu A_\mu - b \phi)
\]

(18)
where the trace is taken over the fundamental representation of the group $SU(N)$ for definitiveness. With $t^a$ being the generators of $SU(N)$, the trace is normalized by $Tr(t^at^b) = \frac{1}{2}\delta^{ab}$. The covariant derivative is, in accordance with the shift $A\rightarrow A + \epsilon\partial A$, defined by $F_\mu \equiv \partial_\mu + g[A_\mu]$. To avoid unnecessary notational inferences, we have restricted ourselves to a manifold with flat metric $g_{\mu\nu}$. Otherwise, although the theory is still generally covariant, explicit $detg$ factors appear in the gauge-fixing terms.

The full action (15) is of course invariant under the ordinary BRST symmetry

$$\delta A_\mu = \epsilon D_\mu \bar{c},$$
$$\delta \bar{c} = -\frac{1}{2}\epsilon [c, \bar{c}],$$
$$\delta \bar{c} = \epsilon \bar{c},$$
$$\delta \bar{c} = 0$$

as well as the anti-BRST symmetry

$$\delta A_\mu = \epsilon D_\mu \bar{c},$$
$$\delta \bar{c} = -\frac{1}{2}\epsilon [\bar{c}, c],$$
$$\delta \bar{c} = \epsilon \bar{c},$$
$$\delta \bar{c} = 0$$

The surprising observation of ref.3 is that in addition the action (13) is found to be invariant under the following transformations:

$$\delta A_\mu = -\epsilon_{\mu\nu\rho} \epsilon^\rho c,$$
$$\delta c = 0,$$
$$\delta \bar{c} = \epsilon A_\mu,$$
$$\delta \bar{c} = -\epsilon D_\mu c$$

and a similar set of transformations corresponding to the anti-BRST invariance[9]:

$$\delta A_\mu = -\epsilon_{\mu\nu\rho} \epsilon^\rho \bar{c},$$
$$\delta \bar{c} = \epsilon A_\mu,$$
$$\delta \bar{c} = 0,$$
$$\delta \bar{c} = \epsilon D_\mu \bar{c}$$

These transformations involve anticommuting vector parameters $\epsilon_c$ and $\epsilon_{\bar{c}}$, and one is naturally led to the idea that this new invariance could be related to the IOStSp(3|2) algebra, either in its usual form, or in the modified form discussed earlier. If true, the only possibility would be to identify the generator of (21) with $J_\mu$, and that of (22) with $J_{-\mu}$ (the choice of $+$ or $-$ being a matter of convention). To see if this first step in identifying the algebra is a consistent assumption, we first compute the anticommutator $[J_{\mu\nu}, J_{-\mu\nu}]$ using the above identification, and indeed we find that $[\delta, \delta'] [A_\mu, \bar{c}, c, b] = \epsilon_{\rho\sigma\tau} \epsilon^{\rho\sigma} D_\tau [A_\mu, \bar{c}, c, b]$, which precisely corresponds to eq.(12). This holds only when one uses the equations of motion, so the symmetry can at most be realized on-shell.

Next, we identify $P_\mu$ with the BRST and anti-BRST generators. In this respect we differ from related work on IOSp symmetries by Siegel and Zwiebach[10], who instead identify the BRST generators with a light cone component of $J_{+\mu}$. Now all other (anti)commutation relations of eqs.(1)-(13) involving $J_{+\mu}, J_{-\mu}, P^\mu$ and $P^a$ are easily seen to be satisfied as well. In a few instances the equations of motion must be used.

It now only remains to identify the generators $J_{\mu\nu}$. Since they involve rotations in the anticommuting space of coordinates and refer only to the theory after gauge fixing, we expect them to act trivially on the $A_\mu$ field. Ghost number counting gives us yet another hint of what kind of symmetry it should be, since the ghost number of $c, \bar{c}$ and $b$ have already been fixed (at -1,1 and 0, respectively). Combining this with the constraints imposed by the scaling dimensions of these fields, we find after a little experimentation:

$$\Delta A_\mu = 0, \quad \Delta c = 0, \quad \Delta \bar{c} = 0$$

as well as

$$\Delta A_\mu = 0, \quad \Delta c = 0, \quad \Delta \bar{c} = 0$$

The commutator of these transformations yields yet another symmetry, which turns out to be generated by the ghost number charge $Q_c$:

$$\Delta A_\mu = 0, \quad \Delta c = 0, \quad \Delta \bar{c} = 0$$

We identify the generator of (23) with $J_{+++}$, the generator of (24) with $J_{---}$, and finally the generator of (25) with $J_{+-}$. In fact, one could have guessed this from the beginning, since these generators simply satisfy the $Sp(2)$ algebra.

$$[\sigma^+, \sigma^-] = i\sigma^0; \quad [\sigma^0, \sigma^\pm] = \pm 2\sigma^\pm$$

of general gauge theories in Landau gauge[13].

Actually, Landau gauge seems to be rather special also at this point. As one can easily check, there is a different invariance:

$$\Delta A_\mu = 0, \quad \Delta c = 0, \quad \Delta \bar{c} = 0$$

which precisely corresponds to eq.(12). This holds only when one uses the equations of motion, so the symmetry can at most be realized on-shell.
and the related anti-BRST type symmetry obtained basically by replacing the ghost by its antighost.

Note that this "equation of motion symmetry" (it reduces to the identity on-shell) holds for any Yang-Mills theory gauge fixed to Landau gauge, in any number of dimensions. There is clearly an infinite set of such symmetries, all reducing to the identity on shell. Very similar kinds of equation of motion symmetries have been found for particle, superparticle and string actions. However, the generators of this type of symmetry cannot be identified with $J_{AB}$.

It is now straightforward to confirm that including the former identifications of $J_{AB}$ (eqs.(23)-(25)), we indeed have a full set of generators of the super Lie algebra defined by eqs. (11)-(11), with the modifications (12) and (19). To summarize, the identifications are the following:

$$
\begin{align*}
\delta_+ &\rightarrow \delta \\
\delta_- &\rightarrow \partial_+ \\
\Theta &\rightarrow \delta' \\
J &\rightarrow \Delta \\
J &\rightarrow \tilde{\lambda} \\
J &\rightarrow \tilde{\lambda}.
\end{align*}
$$

A number of questions obviously still remain to be answered. What is the deeper reason for the appearance of this large symmetry group for this particular theory, and this particular gauge? Can it be extended to other theories or to other gauges? Can it be made to close off-shell by additional auxiliary fields (without changing the theory)? It would also be interesting to understand the consequences of this extra symmetry. Is it perhaps responsible for the perturbative infrared-finiteness of Chern-Simons theory in Landau gauge?

In fact very similar symmetries do appear in other gauge theories as well. As a trivial example, consider pure (free) QED given by the action

$$S = \int d^4x \left( \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \partial \delta' \phi + \partial \phi A^\mu + \frac{1}{2} \phi \partial^2 \right)$$

Apart from the usual BRST and anti-BRST invariance this theory in $\alpha = -1$ gauge is also invariant under

$$\begin{align*}
\delta A_\mu &= -\epsilon_\mu c \\
\delta \phi &= 0 \\
\delta A^\mu &= \epsilon_\mu \phi \\
\delta \phi &= -\epsilon_\mu \partial c
\end{align*}$$

and a similar symmetry with the ghost replaced by the antighost. A systematic analysis of such accidental symmetries of Yang-Mills theories in certain gauges, and their consequences in terms of further constraints on the usual Ward Identities, will be published elsewhere.

Finally, one obvious question is whether the extra symmetry of Chern-Simons theory, which has been demonstrated at the classical level only, can be anomalous. There have in fact been indications that this might be the case[8], but a recent analysis[16] has shown that at least the subalgebra (18)-(19) is non-anomalous. It thus appears very unlikely that the full global symmetry group discussed here will be broken by quantum fluctuations.

Acknowledgment:

P.H.D would like to thank the Department of Physics at the University of São Paulo for the hospitality extended to him at the time when this work was initiated.

---

1 See also the recent study of scale invariance of this theory[16].
Referências


