LEFT-RIGHT ASYMMETRY AND MINIMAL COUPLING

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ABSTRACT

We extend, to arbitrary spin massless particles, the usual chiral components associated to massless spin $\frac{1}{2}$ particles. In the case of spin 1 particles the nonzero generalized helicity components satisfies Weyl's equations and are associated to observables (Electric and Magnetic Fields) whereas the zero generalized helicity components are related to non-observables (Electromagnetic Potentials). We show that nature, as described by QED, is asymmetric with regard to left and right since matter couples only to some chiral components of the zero mass spin 1 field. In this way we have shown that the minimal coupling substitution is just a physical consequence of the left–right asymmetry of nature.

1 – INTRODUCTION

For massless spin $\frac{1}{2}$ particles one can define chiral components $\psi_L$ and $\psi_R$ of a basic field $\psi$ as follows:

\[
\psi_R = \frac{1}{2} (I + \gamma^5) \psi
\]

\[
\psi_L = \frac{1}{2} (I - \gamma^5) \psi
\]

It is an intriguing aspect of the fundamental interactions that, at least at low energies, the right and left components interact in a different way with ordinary matter. As a matter of fact there is no evidence at all that the right hand component interacts with ordinary matter.

Nature, at low energies, is definitely asymmetric with regard to left and right. The first clue in this direction was given by the old V–A theory of the weak interactions$^1$. The empirical evidence for this asymmetry comes from the breakdown of space reflection invariance by the weak interaction$^2$.

This paper deals with the question on whether there are other evidences in nature of an asymmetric coupling of chiral components of zero mass particles to ordinary matter. Besides the neutrinos the next, nontrivial, zero mass particles that couples to ordinary particles are the photons. The question that we adress ourselves in this paper is whether the coupling of photons reflects some kind of asymmetry between the coupling of chiral components to ordinary matter. We find that this is indeed the case that minimal coupling is nothing but a reflex of this asymmetry.

Obviously one needs to specify what we mean by chiral components of a spin zero field. These components have a well defined meaning if we work within the Bargmann Wigner (B.W.) method. If $\psi$ represents the B.W. field associated to a spin 1 particle than one can define generalized chiral components for the field as follows:
\[ \psi_{RR} = \frac{1}{2} (I + \gamma^5) \otimes \frac{1}{2} (I + \gamma^3) \psi \]
\[ \psi_{LL} = \frac{1}{2} (I - \gamma^5) \otimes \frac{1}{2} (I - \gamma^3) \psi \]
\[ \psi_{RL} = \frac{1}{2} (I - \gamma^5) \otimes \frac{1}{2} (I + \gamma^3) \psi \]
\[ \psi_{LR} = \frac{1}{2} (I + \gamma^5) \otimes \frac{1}{2} (I - \gamma^3) \psi \]

We will show that QED can be completely formulated by taking only the interaction of \( \psi_{LR} \) and \( \psi_{RL} \) with ordinary matter fields. Within the context of QED, we will see that the usual minimal substitution prescription \( \partial \rightarrow \partial - i e A \) is nothing but a reflex of the left–right asymmetry in the case of spin 1 particles.

The plan of this paper is the following:

In chapter II we extend to particle of arbitrary spin \( s \), the usual spin \( \frac{1}{2} \) chiral components. This extension is possible in the context of B.W. theory. The totally right (left) components have generalized helicity components \( s \) (\( -s \)) and obey the generalized Weyl's equation.

As an exercise, and in order to illustrate how the B.W. theory works, we present in chapter III the B.W. theory for spin 1 massive particles. The interesting point here is that clearly B.W. theory leads to a complete description of massive particles by associating to these particles a symmetric rank 2 tensor field instead of associating particles to a rank 1 tensor (the usual procedure). The subsidiary condition \( \mathcal{J}^\mu B_\mu = 0 \), for instance, follows naturally from the decomposition of the basic B.W. field into the spinor space and the B.W. equation.

In chapter IV we make the extension of the B.W. theory to massless spin 1 particles. In this case it is straightforward to show that the two components of the chiral components are associated to the potential whereas the other components are associated to the potentials.

In chapter V we formulate QED in terms of the chiral components \( \psi_{RR}, \psi_{RL}, \psi_{LR} \) and \( \psi_{LL} \). Here we show explicitly that one can formulate QED as long as matter couples only to some components of the chiral fields. That is, QED is manifestly left–right asymmetric.

In chapter VI we touch on the question of the quantization of the B.W. fields. This is achieved by imposing appropriate commutation relation among the B.W. components.

We end this paper with a chapter dedicated to conclusions.
II - THE BASIC FRAMEWORK

In this chapter we will describe the general framework for studying massive and massless particles of arbitrary spin.

Our starting point for describing massive particles is Bargmann Wigner (B.W.) theory\(^3\). In the case of massless particles we define generalized chiral components that are an extension of particles of any spin of the usual ones for spin \(\frac{1}{2}\) particles. Some of these components satisfy the usual Weyl's equations, the others satisfy equations analogous to Weyl's equations.

Remembering that a rank one spinor transforms, under Poincaré transformation as

\[ \eta(x) \rightarrow \eta(x') = D(\ell) \eta(x) \]

where, in order to ensure the relativistic invariance of Dirac's equations one has to require

\[ D^+(\ell) \gamma^\mu D(\ell) = \delta^\mu_\nu \gamma^\nu \]

then, a 2s rank spinor \( \psi(x) \) transforms as

\[ \psi(x) \rightarrow \psi'(x') = D(\ell) \otimes \cdots \otimes D(\ell) \psi(x) \]

Analogously, the 2s rank spinor \( \bar{\psi}(x) \) defined by

\[ \bar{\psi}(x) = \gamma^\mu \psi(x) \gamma^\nu \otimes \cdots \otimes \gamma^\nu 
\]

transforms like

\[ \bar{\psi}(x) \rightarrow \bar{\psi}'(x') = \bar{\psi}(x) D^+(\ell) \otimes \cdots \otimes D^+(\ell) \]

where we have used the property that

\[ \gamma^\mu D^+(\ell) \gamma^\nu = D^+(\ell) \]

Throughout this paper we use the notation of Bjorken and Drell book\(^4\). In an appendix of this paper we present some of the matrices and notation that we have used.

Within the BW theory\(^3\) a massive particle of mass \( m \) and spin \( s \) is described by a 2s rank spinor field

\[ \psi_{a_1 \cdots a_{2s}}(x) \quad a_u = 1, 2, 3, 4 \]

that is symmetric in its spin indices, obeying a system of \( 2s \) Dirac equations:

\[ i \gamma^\mu \psi_{a_1 \cdots a_{2s}}(x) = m \psi_{a_1 \cdots a_{2s}}(x) \quad k = 1, 2s \]

or in a different notation

\[ i \partial_0 \otimes I \psi = m \psi \]
\[ I \otimes i \gamma^\mu \psi = m \psi \]
\[ \cdots \]
\[ I \otimes I \otimes \cdots \otimes i \gamma^\nu \psi = m \psi \quad (II.1) \]

The symmetry in the spin indices of the BW spinor field \( \psi \) allows us to express
this field in terms of a linear combination of symmetric matrices of the spinor space. In this expansion, new fields appear (which, as illustrated in the case of spin 1 particles, are the usual spin 1 fields) which are coefficients in the expansion. Some properties for these new fields can be deduced from Dirac equation (II.1) for the BW field.

For the sake of completeness let us write the decomposition of the rank 2 spinor field \( \psi_{a_2 b_2} \) associated to a mass \( m \) and spin 1 particle in terms of the symmetric matrices \( (\gamma^\mu C) \) and \( (\sigma^{\mu\nu} C) \). \( \psi_{a_2 b_2} \) admits the following decomposition:

\[
\psi_{a_2 b_2}(x) = \sqrt{m} \left[ B_{\mu}(x)(\gamma^\mu C)_{a_2 b_2} - \frac{1}{2m} G_{\mu\nu}(x)(\sigma^{\mu\nu} C)_{a_2 b_2} \right] \quad (II.2)
\]

where the field \( B_{\mu}(x) \) is a vector field (that as will be shown later in the usual spin 1 field), and \( G_{\mu\nu}(x) \) is a field not yet determined and that, in principle should involve derivatives of the \( B_{\mu}(x) \) field. This is actually what happens. As will be shown in the next section, from (II.1) and (II.2) it follows that

\[
G_{\mu\nu}(x) = \partial_{\nu} B_{\mu}(x) - \partial_{\mu} B_{\nu}(x) \quad (II.3)
\]

In the case of massless spin \( \frac{1}{2} \) particles, one starts with Weyl's equation. In this case the basic set of equations is:

\[
\begin{align*}
\left\{-i\sigma^\mu \partial_\mu - i\phi \cdot \partial_\mu \right\} \xi(x) &= 0 \\
\left\{-i\sigma^\mu \partial_\mu + i\phi \cdot \partial_\mu \right\} \chi(x) &= 0
\end{align*}
\]

(II.4)

where \( \xi(x) \) and \( \chi(x) \) are two component spinors. These spinors are eigenstates of the helicity operator \( \frac{1}{2} \phi \cdot \partial \left[ \hat{n} = \frac{\hat{p}}{|\hat{p}|} \right] \) with eigenvalues \( \pm \frac{1}{2} \):

\[
\begin{align*}
\frac{1}{2} (\partial \cdot \phi) \xi &= \frac{1}{2} \xi \\
\frac{1}{2} (\partial \cdot \phi) \chi &= -\frac{1}{2} \chi 
\end{align*}
\]

(II.5)

If one write a four component spinor as

\[
\hat{\phi} = \begin{bmatrix} \xi \\ \chi \end{bmatrix}
\]

then one can define, as usual, the right and left components as

\[
\begin{align*}
\hat{\phi}_R &= \frac{1}{2} (I + \gamma^5) \hat{\phi} \\
\hat{\phi}_L &= \frac{1}{2} (I - \gamma^5) \hat{\phi}
\end{align*}
\]

(II.6)

With the definition (II.6) one can then show that Weyl's equations (II.4) are equivalent to the equations:

\[
\begin{align*}
\left\{-i\sigma_{\mu\nu} \partial^{\mu\nu} - i\phi \cdot \partial_\mu \right\} \xi_R &= 0 \\
\left\{-i\sigma_{\mu\nu} \partial^{\mu\nu} + i\phi \cdot \partial_\mu \right\} \chi_L &= 0
\end{align*}
\]

(II.6)

For particles of arbitrary spin we assume (like in BW's theory for massive particles) that a particle of spin \( s \), without mass is described by a spinor

\[
\hat{\phi}_{a_1 a_2 \ldots a_{2s}}(x) \quad a_i = 1, 2, 3, 4
\]

of rank 2s, symmetric in its spin variables, which obeys a system of 2s Dirac's equations:

\[
i \partial_{\lambda^k} \hat{\phi}_{a_1 a_2 \ldots a_{2s}}(x) = 0 \quad \text{with} \quad k = 1, 2s
\]

(II.7)
The definition of chiral components for massless particles of arbitrary spin is:

\[ \tilde{\psi}_{\text{R...R}}(x) \equiv \tilde{\psi}_{R_1 \ldots R_{2s}}(x) = \frac{1}{2} (I^+ - \gamma^2)_{\alpha_1 \beta_1}^\dagger \frac{1}{2} (I^+ - \gamma^2)_{\alpha_2 \beta_2} \ldots \frac{1}{2} (I^+ - \gamma^2)_{\alpha_{2s} \beta_{2s}}^\dagger \tilde{\psi}_{\alpha_1 \ldots \alpha_{2s}}(x) \]

\[ \tilde{\psi}_{\text{R...RL}}(x) \equiv \tilde{\psi}_{R_1 \ldots R_{2s-1} \text{L}_{2s}}(x) = \frac{1}{2} (I-\gamma^2)_{\alpha_1 \beta_1} \frac{1}{2} (I-\gamma^2)_{\alpha_2 \beta_2} \ldots \frac{1}{2} (I-\gamma^2)_{\alpha_{2s-1} \beta_{2s-1}} \tilde{\psi}_{\alpha_1 \ldots \alpha_{2s-1} \beta_{2s}}(x) \]

\[ \tilde{\psi}_{\text{L...L}}(x) \equiv \tilde{\psi}_{\text{L}_{2s} \ldots \text{L}_{2s}}(x) = \frac{1}{2} (I-\gamma^2)_{\alpha_1 \beta_1} \frac{1}{2} (I-\gamma^2)_{\alpha_2 \beta_2} \ldots \frac{1}{2} (I-\gamma^2)_{\alpha_{2s} \beta_{2s}}^\dagger \tilde{\psi}_{\alpha_1 \ldots \alpha_{2s}}(x) \quad \text{(II.8)} \]

From the equation (II.7) we may derive the following equations:

\[ i\slashed{D}_{\alpha_1} \tilde{\psi}_{R_1 \ldots R_{2s}}(x) = 0 \]

\[ i\slashed{D}_{\alpha_1} \tilde{\psi}_{R_1 \ldots R_{2s-1} \text{L}_{2s}}(x) = 0 \]

\[ i\slashed{D}_{\alpha_1} \tilde{\psi}_{\text{L}_{2s} \ldots \text{L}_{2s}}(x) = 0 \quad \text{(II.9)} \]

If we adopt the following notation:

\[ \tilde{\psi}_{\text{R...R}} \sim \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \phi \begin{pmatrix} \zeta_{\text{HS}} \end{pmatrix} = \begin{pmatrix} \xi_{\text{HS}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[ \tilde{\psi}_{\text{R...RL}} \sim \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \phi \begin{pmatrix} \zeta_{\text{HS}-1} \\ \zeta_{\text{HS}} \end{pmatrix} = \begin{pmatrix} \xi_{\text{HS}-1} \\ 0 \end{pmatrix} \phi \begin{pmatrix} 0 \\ \zeta_{\text{HS}} \end{pmatrix} \]

\[ \tilde{\psi}_{\text{L...L}} \sim \begin{pmatrix} 0 \\ \zeta_{\text{HS}} \end{pmatrix} \phi \begin{pmatrix} 0 \\ \zeta_{\text{HS}} \end{pmatrix} = \begin{pmatrix} 0 \\ \zeta_{\text{HS}} \end{pmatrix} \]

\[ \tilde{\psi}_{\text{R...R}} \sim \begin{pmatrix} \phi \\ 0 \\ 0 \end{pmatrix} \]

\[ \tilde{\psi}_{\text{R...RL}} \sim \begin{pmatrix} 0 \\ \phi \\ 0 \end{pmatrix} \]

\[ \tilde{\psi}_{\text{L...L}} \sim \begin{pmatrix} 0 \\ 0 \\ \phi \end{pmatrix} \quad \text{(II.11)} \]

It is clear that if \( \tilde{\psi} \) transforms under the Poincare group like the tensor product of 2s bispinor\(^5\): 

\[ \tilde{\psi} \sim \begin{pmatrix} \xi_1 \\ \zeta_{\text{HS}} \end{pmatrix} \phi \begin{pmatrix} \xi_{\text{HS}} \end{pmatrix} \]

then, in chiral representation, representation that we will use from now on in the massless case, \( \tilde{\psi}_{\text{R...R}}, \tilde{\psi}_{\text{R...RL}}, \ldots, \tilde{\psi}_{\text{L...L}} \), will transform like:
Where \( \varphi, \varphi', \ldots, \varphi^{(2s-1)} \), are respectively the only non zero components of \( \tilde{\psi}_{R\ldots R} \), \( \tilde{\psi}_{R\ldots R_L} \), \( \ldots \), \( \tilde{\psi}_{L\ldots L} \) and are two component spinors of rank \( 2s \):

\[
\varphi = \varphi_{b_1 b_2 \ldots b_{2s}}
\]

\[
\varphi' = \varphi'_{b_1 b_2 \ldots b_{2s}}
\]

\[
\vdots
\]

\[
\varphi^{(2s-1)} = \varphi^{(2s-1)}_{b_1 b_2 \ldots b_{2s}}
\]

Using Eqs. (II.9) we can see that \( \varphi \) and \( \varphi^{(2s-1)} \) satisfies Weyl's equations (II.12):

\[
(-i\sigma^0 \partial_0 - i\vec{\sigma} \cdot \vec{\theta}) \otimes 1 \otimes 1 \otimes 1 \varphi(x) = 0
\]

(II.12a)

\[
(-i\sigma^0 \partial_0 + i\vec{\sigma} \cdot \vec{\theta}) \otimes 1 \otimes 1 \otimes 1 \varphi^{(2s-1)}(x) = 0
\]

(II.12b)

where \( \varphi \) and \( \varphi^{(2s-1)} \) are symmetric two component spinors of rank \( 2s \).

Eq. (II.12b) can be derived from (II.12a) by space reflection, so one can consider just one of the Eq. (II.12a) or (II.12b).

The other components \( \varphi', \ldots, \varphi^{(2s-1)} \) satisfy equations analogous to Weyl's.

III - B.W. THEORY FOR MASSIVE SPIN 1 PARTICLES

This chapter is introduced in order to show how one works with the B.W. theory to describe spin 1 particles. We will see, in an explicit way, that the method is, formally, richer than the usual one in the sense that it allows us to write equations in terms of "observables" (Fields analogous to \( E \) and \( H \) in Maxwell's theory) and leads in a straightforward way to the subsidiary condition \( \partial^\mu B_\mu = 0 \).

Although the method is formally better it is just a little bit trickier, as we will see, to get the physics out of B.W. method.

Free field

Within the B.W. method a spin 1 massive particle of mass \( m \) is described, in the non interacting case, by a rank 2 symmetric spinor \( \psi_{a b}(x) \) obeying a system of two Dirac type equations:

\[
(i\sigma^\mu \partial_\mu) \psi = m\psi
\]

(III.1)

\[
(1 \otimes i\theta) \psi = m\psi
\]

Equations (III.1) may be derived from the following lagrangian\(^6\):

\[
L_0 = \bar{\psi} \left\{ \frac{i}{2} \left[ \gamma^\mu \partial_\mu + i \gamma^\mu \right] \frac{1}{2} \right\} \psi - m \bar{1} \otimes 1 \psi
\]

(III.2)

If we treat the field \( \psi_{a b} \) as the independent variable, one gets in particular:

\[
[i \mathcal{S} - m]_{a a'} \psi_{a b}(x) = 0
\]

(III.3)
We replace \( \psi_{\alpha_1 \alpha_2} \) in (III.3) by its decomposition (II.2) and obtain

\[ i \varepsilon_{\mu_1 \mu_2}^\alpha \partial_\alpha \left\{ B_\mu (x) (\gamma^\rho C)_{\alpha_1 \alpha_2} - \frac{1}{2m} G_{\mu \nu} (x) (\sigma^{\mu \nu} C)_{\alpha_1 \alpha_2} \right\} \]

\[ = \hbar \left\{ B_\mu (x) (\gamma^\rho C)_{\alpha_1 \alpha_2} - \frac{1}{2m} G_{\mu \nu} (x) (\sigma^{\mu \nu} C)_{\alpha_1 \alpha_2} \right\}. \]  

(III.4)

In order to see that (III.4) leads to the usual equations one has to make some simple operations involving \( \gamma \) matrices. For instance, if one multiplies (III.4) by \((C^{-1})_{\beta \alpha}^\lambda\) and sums over \( \alpha_1 \alpha_2 \) one gets:

\[ i \partial_\alpha B_\mu \text{Tr}(\gamma^\alpha \gamma^\mu) = 0 \]

from which it follows that:

\[ \partial_\mu B^\mu = 0. \]  

(III.5)

It is interesting to see that the subsidiary condition \( \partial_\mu B^\mu = 0 \) follows directly from BW equation. If, on the other hand, one multiplies (III.4) by \((C^{-1})_{\beta \alpha}^\lambda\) one gets:

\[ -\frac{i}{2m} \partial_\alpha G_{\mu \nu} \text{Tr}(\gamma^\alpha \sigma^{\mu \nu} \gamma^\lambda) = m B_\mu \text{Tr}(\gamma^\mu \gamma^\lambda) \]

from which one gets

\[ -\partial^\mu G_{\mu \nu} + m^2 B_\nu = 0. \]  

(III.6)

Finally if we multiply (III.4) by \((C^{-1})_{\beta \alpha}^\lambda\) one gets:

\[ i \partial_\alpha B_\mu \text{Tr}(\gamma^\alpha \gamma^\beta \gamma^\lambda) = \frac{1}{2m} G_{\mu \nu} \text{Tr}(\sigma^{\mu \nu} \gamma^\lambda). \]

from which it follows that:

\[ \partial_\alpha B_\mu \partial^\lambda + \partial^\lambda B_\mu \partial^\beta - \partial^\beta B^\lambda = G^{\beta \lambda}. \]

(III.7)

By using (III.5) in (III.7) one gets

\[ G^{\beta \lambda} = \partial^\lambda B^\beta - \partial^\beta B^\lambda \]

(III.8)

that is, in the decomposition (II.2) of the field \( v \), the only acceptable tensor is \( \delta^\rho B^\mu - \delta^\mu B^\rho \).

We have then seen that BW equations leads to the following restrictions upon the fields \( B^\mu \) and \( G^{\mu \nu}:

\[ \delta^\mu B^\mu = 0 \]

\[ G_{\mu \nu} = \delta^\mu B_\nu - \delta^\nu B_\mu \]  

(III.9)

\[ -\partial^\mu G_{\mu \nu} + m^2 B_\nu = 0. \]

In order to see the equivalence between the BW method and the usual approach, in which we assign to a vector field \( B_\mu \) a spin 1 particle, let us now write \( L_0 \) in terms of \( B_\mu \).

By using decomposition (II.2) one gets:

\[ L_0 = 4m^2 B^\mu B_\mu - 2 G^{\mu \nu} G_{\mu \nu} \]  

(III.10)

or also using (III.8)

\[ L_0 = 4m^2 B^\mu B_\mu + 4 \partial^\nu B^{\nu \rho}(\partial_\rho B_\mu - \partial_\mu B_\rho). \]  

(III.11)
If we consider $B_\mu$ as the independent field then one gets from (III.11) the usual
Euler–Lagrange for the $B_\mu$ field that is:

$$m^2 B_\nu - \partial^\mu G_{\mu\nu} = 0 \quad (III.12)$$

where

$$G_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu.$$  

Eq. (III.12) is the same that eq. (III.6).

**Interacting fields**

Let us consider the interaction of massive spin 1 particles with massive spin $\frac{1}{2}$
particles, described, as usual, by rank 1 spinor field $\eta$.

If we restrict ourselves to lagrangians that are linear in the $\psi$ fields (that leads,
ultimately, to renormalizable models) then, the forms that are compatible with Lorentz
invariance assume the form (III.13):

$$\mathcal{L}_{\text{int}} = \bar{\eta}_s \psi_s \bar{\psi}_s \eta_s + \bar{\eta}_s \psi_s \bar{\psi}_s \eta_s + \text{h.c.} = \bar{\eta} \psi \eta + \bar{\eta} \psi \eta + \text{h.c.} \quad (III.13)$$

where $\bar{\eta}$ and $\eta$ are constants with dimension $[L]^{1/2} = \frac{1}{[M]^{1/2}}$.

In the following we shall take $g_2 = 0$. At least in the zero mass limit nature
prefers this type of coupling.

We shall study the following total lagrangean for a spin 1 massive field interacting
with a spin $\frac{1}{2}$ massive field:

$$\mathcal{L} = \bar{\psi} \left( I \partial^\mu \gamma^\mu + \frac{1}{2} \gamma^\mu \partial^\mu \right) \psi + \bar{\eta} \psi \gamma^\nu \partial_\nu \eta + \bar{\eta} \eta \gamma^\nu \partial_\nu \eta \quad (III.14)$$

Replacing $\psi$ by its decomposition (II.2) we obtain for $\mathcal{L}$

$$\mathcal{L} = 4m^2 B^\mu \partial_\mu B_\nu - 2 G^{\mu\nu} \partial_\mu \eta - \frac{1}{2m} \eta \gamma^\mu \partial_\mu \eta + \bar{\eta} \eta (\partial_\mu - m_1) \eta \quad (III.15)$$

Using $G^{\mu\nu} = \partial^\mu B_\nu - \partial^\nu B_\mu$ we can write $\mathcal{L}$ in the following form

$$\mathcal{L}_0 = 4m^2 B^\mu B_\mu + 4 \partial^\mu B^\nu (\partial_\mu B_\nu - \partial_\nu B_\mu) -$$

$$- \frac{1}{2m} \eta \gamma^\mu \partial_\mu \eta + \frac{1}{m_1} \bar{\eta} \partial_\mu \eta \gamma^\mu \eta \right) + \bar{\eta} (\partial_\mu - m_1) \eta \quad (III.16)$$

Writing now Lagrange's equations with respect to $B^\nu$ and $\eta$

$$\frac{\partial \mathcal{L}}{\partial B^\nu} - \partial^\nu \left[ \frac{\partial \mathcal{L}}{\partial (\partial^\nu B^\nu)} \right] = 0 \quad (III.17)$$

we obtain from (III.16)

$$4m^2 B_\nu - \frac{1}{2m} \eta \gamma_\nu \eta - \bar{\eta} \left( \frac{1}{2m} \eta \gamma_\nu \eta \right) = 0$$

or also remembering that $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

$$\bar{\eta} \left( \partial_\mu - m_1 \right) \gamma^\mu \partial_\nu \eta - \frac{1}{2m} \eta \gamma^\mu \partial_\nu \eta$$

or

$$\bar{\eta} \left( \partial_\mu - m_1 \right) \gamma^\mu \partial_\nu \eta - \frac{1}{2m} \eta \gamma^\mu \partial_\nu \eta$$
On the other hand, we obtain from (III.17)

\[ \left\{ i \partial_\tau - m_2 \right\} \eta_{a_2} = 2 \xi_1 \sqrt{m} \left\{ \gamma_\mu \partial_r - \frac{1}{2m} G_{\mu \nu} \delta^{\mu \nu} \right\} \eta_{a_2} \eta_{a_2}. \]  

(III.19)

Treating now \( \eta_{a_2} \) and \( \bar{\eta}_{a_2} \) as independent fields we obtain using (III.9) and Lagrange equations:

\[ \frac{\partial \mathcal{L}}{\partial \eta_{a_2}} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\eta}_{a_2}} = 0 \]

\[ \frac{\partial \mathcal{L}}{\partial \bar{\eta}_{a_2}} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\bar{\eta}}_{a_2}} = 0 \]

the following equations

\[ \frac{i}{2} \left\{ \left[ \bar{\eta}_{a_2} \right] \delta_{a_2} + \delta_{a_2} \bar{\eta}_{a_2} \right\} \partial_\mu \eta_{a_2} - m_2 \eta_{a_2} + \frac{g_1}{\sqrt{m}} \bar{\eta}_{a_2} \eta_{a_2} = 0 \]  

(III.20)

\[ 2 \xi_1 \sqrt{m} \bar{\eta}_{a_2} - m_2 \bar{\eta}_{a_2} - i \partial_\mu \left( \bar{\eta}_{a_2} \right) \bar{\eta}_{a_2} = 0. \]  

(III.21)

Replacing \( \eta_{a_2} \) by its decomposition (II.2) in (III.20) we obtain

\[ 0 = \frac{i}{2} \partial_\mu \left\{ B_\mu \left( \gamma^\mu \gamma^\nu \eta_{a_2} \gamma_{a_2} \right) + \frac{1}{2m} G_{\mu \nu} \left( \gamma^\mu \gamma^\nu \eta_{a_2} \right) \right\} \]

\[ - \frac{i}{2} \partial_\mu \left\{ B_\mu \left( \gamma^\mu \gamma^\nu \eta_{a_2} \gamma_{a_2} \right) + \frac{1}{2m} G_{\mu \nu} \left( \gamma^\mu \gamma^\nu \eta_{a_2} \right) \right\} \]

\[ - m \left\{ B_\mu \left( \gamma^\mu \eta_{a_2} \gamma_{a_2} \right) + \frac{g_1}{\sqrt{m}} \bar{\eta}_{a_2} \right\}. \]  

(III.22)

Now multiplying (III.22) by \( (\gamma^\rho)^{\mu \nu} \) we obtain

\[ \partial_\mu G^{\rho \mu} = - m_2 \delta^\rho + \frac{g_1}{2} \eta \gamma^\rho \eta \]  

(III.23)

Model (III.23) leads to equations analogous to Maxwell's. In fact, in analogy with massless case, we name

\[ G^{\alpha k} = \xi^k \]  

(III.24)

\[ G^{(1,0)} = \varepsilon^{\mu \nu} \]  

we obtain from (III.23) the following set of equations:

\[ \hat{\psi} \cdot \vec{E} + m_2 \Psi = g_1 \frac{\sqrt{m}}{2} \eta \gamma^\rho \eta \]

\[ \partial \vec{E} - \hat{\psi} \times \vec{H} - m_2 \hat{\psi} = - g_1 \frac{\sqrt{m}}{2} \eta \gamma^\rho \eta \]  

(III.25)

\[ \hat{\psi} \cdot \vec{H} = 0 \]

\[ \partial \vec{H} + \hat{\psi} \times \vec{E} = 0. \]
Eqs. (III.25) are analogous to Maxwell's. That is we get a set of coupled equations of first order in terms of the observables $\bar{E}$ and $\bar{H}$.

Finally eq. (III.21) gives rise, after taking the complex conjugate and replace $\psi$ by its decomposition (II.2), to (III.19).

Hamiltonian

We take the form (III.14) of $\mathcal{L}$ as a starting point. The two fields $\eta$ and $\psi$ are independent. We construct conjugate momenta from $\mathcal{L}$ by the standard prescription; so we obtain

$$\tau_{a_1 a_2} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{a_1 a_2}} = \bar{\psi}_{a_1 a_2} \frac{1}{2} \left\{ \gamma_{a_1} \gamma_{a_2} \rho_{a_1 a_2} + \delta_{a_1} \gamma_{a_2} \gamma_{a_2} \right\}$$

(III.26)

$$\tau_{a_1} = \frac{\partial \mathcal{L}}{\partial \dot{\eta}_{a_1}} = \eta_{a_1} i \gamma_{a_1}$$

The Hamiltonian is defined by

$$H = \tau_{a_1 a_2} \dot{\phi}_{a_1 a_2} + \tau_{a_1} \dot{\eta}_{a_1} - \mathcal{L}$$

Replacing $\phi_{a_1 a_2} (\bar{\psi}_{a_1 a_2})$ by its decomposition (II.2) one gets

$$H = 4 \left\{ G^{a \mu} \partial_\mu B_\nu - B^a \partial_\mu G^{a \mu} \right\} + \eta \left( i \gamma^\mu \partial_\mu \eta - 4 \left( m^2 B^{a \mu} B_\mu - \frac{1}{2} G^{a \mu \nu} G_{\mu \nu} \right) + \right.$$  

$$+ \kappa \sqrt{n} \left( B^a \eta \gamma^\mu \eta - \frac{1}{2m} G^{a \mu \nu} \eta \sigma^{\mu \nu} \eta \right) - \eta \left[ i \gamma^\mu \partial_\mu - m \right] \eta$$

(III.27)

Or, integrating by parts:

$$H = 4 \left\{ G^{a \mu} \partial_\mu B_\nu - B^a \partial_\mu G^{a \mu} \right\} - 4 \left\{ m^2 B^{a \mu} B_\mu - \frac{1}{2} G^{a \mu \nu} G_{\mu \nu} \right\} +$$

$$+ \kappa \sqrt{n} \left( B^a \eta \gamma^\mu \eta - \frac{1}{2m} G^{a \mu \nu} \eta \sigma^{\mu \nu} \eta \right) - \eta \left[ i \gamma^\mu \partial_\mu - m \right] \eta$$

(III.29)

where

$$k, j = 1, 2, 3$$

$$\mu, \nu = 0, 1, 2, 3$$

Now, following Bjorken we adopt the notation

$$E^i_j = E^i_j + E^j_i = - \partial_1 B^0 - \partial_0 B^1$$

and we assume that the fields $\bar{E}$ and $\bar{H}$ are real, so we obtain

$$H = 4 \left\{ H^a + B_0 \right\} - 4 \bar{E}^2 - 4 \bar{m}^2 B^a B_0 +$$

$$+ \kappa \sqrt{n} \left( B^a \eta \gamma^\mu \eta + B_\mu \eta \gamma^\mu \eta - \frac{1}{2m} G^{a \mu \nu} \eta \sigma^{\mu \nu} \eta \right) - \eta \left[ i \gamma^\mu \partial_\mu - m \right] \eta$$

(III.27)

$$k, j = 1, 2, 3$$

$$\mu, \nu = 0, 1, 2, 3$$
IV - SPIN 1 MASSLESS PARTICLES

The extension of B.W. theory to massless particles is not straightforward. For example the zero mass limit of the decomposition (II.2) is meaningless. Furthermore, for massless spin 1 particles, the appropriate Lagrangian is not the zero mass limit of (III.2). The appropriate lagrangian in this case is

\[ \mathcal{L}_0 = \bar{\psi} \left\{ \frac{1}{2} \left[ i \gamma^\mu \partial_{\mu} + i \gamma^\rho \partial_{\rho} \right] \right\} \psi . \]  

(IV.1)

As pointed out in chapter II, for massless spin 1 particles one can work with a rank 2 spinor field \( \psi \) but the only relevant combinations are the chiral components

\[ \begin{align*}
\bar{\psi}_{RR} &= \frac{1}{2} (I + \gamma^5) \otimes \frac{1}{2} (I + \gamma^\ell) \bar{\psi}, \\
\bar{\psi}_{RL} &= \frac{1}{2} (I + \gamma^5) \otimes \frac{1}{2} (I - \gamma^\ell) \bar{\psi}, \\
\bar{\psi}_{LR} &= \frac{1}{2} (I - \gamma^5) \otimes \frac{1}{2} (I + \gamma^\ell) \bar{\psi}, \\
\bar{\psi}_{LL} &= \frac{1}{2} (I - \gamma^5) \otimes \frac{1}{2} (I - \gamma^\ell) \bar{\psi}.
\end{align*} \]

(IV.2)

The equations satisfied by these components are

\[ \begin{align*}
(i \gamma^\rho \partial_{\rho}) \bar{\psi}_{RR} &= 0, \\
(i \gamma^\rho \partial_{\rho}) \bar{\psi}_{RL} &= 0, \\
(i \gamma^\rho \partial_{\rho}) \bar{\psi}_{LR} &= 0, \\
(i \gamma^\rho \partial_{\rho}) \bar{\psi}_{LL} &= 0.
\end{align*} \]

(IV.3)

It is simple to check that the lagrangian that gives rise to these equations is:

\[ \mathcal{L}_0 = \bar{\psi}_{RL} (i \gamma^\rho \partial_{\rho}) \psi_{RR} + \bar{\psi}_{LR} (i \gamma^\rho \partial_{\rho}) \psi_{LL} + \bar{\psi}_{RR} (i \gamma^\rho \partial_{\rho}) \psi_{RL} + \bar{\psi}_{LL} (i \gamma^\rho \partial_{\rho}) \psi_{LR} . \]  

(IV.4)

We will verify that \( \mathcal{L}_0 \) given in (IV.4) is Lorentz invariant.

By a Poincaré transformation of the coordinates, the fields transform as below:

\[ \begin{align*}
\bar{\psi}_{RR}(x) &\rightarrow \bar{\psi}_{RR}^{'}(x^{'}) = D \otimes D \frac{1}{2} (I + \det \ell \gamma^\ell) \otimes \frac{1}{2} (I + \det \ell \gamma^\ell) \bar{\psi}(x), \\
\bar{\psi}_{RL}(x) &\rightarrow \bar{\psi}_{RL}^{'}(x^{'}) = D \otimes D \frac{1}{2} (I + \det \ell \gamma^\ell) \otimes \frac{1}{2} (I - \det \ell \gamma^\ell) \bar{\psi}(x), \\
\bar{\psi}_{LR}(x) &\rightarrow \bar{\psi}_{LR}^{'}(x^{'}) = D \otimes D \frac{1}{2} (I - \det \ell \gamma^\ell) \otimes \frac{1}{2} (I + \det \ell \gamma^\ell) \bar{\psi}(x), \\
\bar{\psi}_{LL}(x) &\rightarrow \bar{\psi}_{LL}^{'}(x^{'}) = D \otimes D \frac{1}{2} (I - \det \ell \gamma^\ell) \otimes \frac{1}{2} (I - \det \ell \gamma^\ell) \bar{\psi}(x).
\end{align*} \]

(IV.5)

So we see that by Poincare coordinate transformation

\[ \begin{align*}
\bar{\psi}_{RL} i \gamma^\rho \partial_{\rho} \psi_{RR}^{'} &\rightarrow \bar{\psi}_{RL}^{'} i \gamma^\rho \partial_{\rho} \psi_{RR}, & \text{if } \det \ell = +1, \\
\bar{\psi}_{LR} i \gamma^\rho \partial_{\rho} \psi_{LL}^{'} &\rightarrow \bar{\psi}_{LR}^{'} i \gamma^\rho \partial_{\rho} \psi_{LL}, & \text{if } \det \ell = -1.
\end{align*} \]

(IV.6)

\[ \bar{\psi}_{RR} i \gamma^\rho \partial_{\rho} \psi_{RL}^{'} \rightarrow \bar{\psi}_{RR}^{'} i \gamma^\rho \partial_{\rho} \psi_{RL}, & \text{if } \det \ell = 1, \\
\bar{\psi}_{LL} i \gamma^\rho \partial_{\rho} \psi_{LR}^{'} &\rightarrow \bar{\psi}_{LL}^{'} i \gamma^\rho \partial_{\rho} \psi_{LR}, & \text{if } \det \ell = -1.
\]

So, \( \mathcal{L}_0 \) is globally Lorentz invariant. On the other hand, it is easy to see, remembering that

\[ \psi = \psi_{RR} + \psi_{RL} + \psi_{LR} + \psi_{LL} \]  

(IV.7)
that the lagrangian (IV.4) is equivalent to

$$\mathcal{L}_0 = \bar{\psi} (i \gamma \cdot J) \psi$$

In fact if one substitutes $\psi$ by (IV.7) then the non zero terms of the above lagrangian are those of (IV.4). It is interesting to note the difference between (IV.1) and (IV.4). Although the difference at first sight seems to be just a matter of symmetry, it is a little bit deeper. The difference is, in fact, chiral invariance. In fact lagrangian (IV.4) is invariant under the chiral transformation

$$\bar{\psi} \rightarrow \bar{\psi}^\prime = e^{-i \theta \gamma^5} \cdot \bar{\psi}$$

whereas in the case of (IV.1) there is no simple extension of the usual chiral transformation to spin 1 particles.

Let us consider now the set of Weyl's equations in the case of spin 1 particles. In this case one writes:

$$\bar{\psi} \sim \left[ \begin{array}{c} \xi^I \\ \chi^I \end{array} \right] \otimes \left[ \begin{array}{c} \xi^{II} \\ \chi^{II} \end{array} \right]$$

so

$$\bar{\psi}_{RR} \sim \left[ \begin{array}{c} \xi^I \\ 0 \end{array} \right] \otimes \left[ \begin{array}{c} \xi^{II} \\ 0 \end{array} \right]$$

$$\bar{\psi}_{RL} \sim \left[ \begin{array}{c} 0 \\ \chi^I \end{array} \right] \otimes \left[ \begin{array}{c} 0 \\ \chi^{II} \end{array} \right]$$

$$\bar{\psi}_{LR} \sim \left[ \begin{array}{c} 0 \\ \chi^I \end{array} \right] \otimes \left[ \begin{array}{c} \xi^{II} \\ 0 \end{array} \right]$$

$$\bar{\psi}_{LL} \sim \left[ \begin{array}{c} 0 \\ \chi^I \end{array} \right] \otimes \left[ \begin{array}{c} 0 \\ \chi^{II} \end{array} \right]$$

Weyl's equations can be derived in this case, by taking only the $\bar{\psi}_{RR}$, $\bar{\psi}_{LL}$ components and are

$$(-i \sigma^i \partial_0 - i \vec{\gamma} \cdot \vec{J}) \chi^{II} = 0$$

$$(-i \sigma^i \partial_0 + i \vec{\gamma} \cdot \vec{J}) \xi^{II} = 0$$

where $\xi^{II}$, $\chi^{II}$ are supposed to be symmetrized in spin indices.

The generalization helicity operator for spin 1 particles is

$$\frac{1}{2} \left( \vec{\Sigma} \cdot \hat{n} + 1 \right) \equiv W$$

It is straightforward to show that $\bar{\psi}_{RR}$, $\bar{\psi}_{RL}$, $\bar{\psi}_{LR}$, $\bar{\psi}_{LL}$ are eigenstates of $W$ with eigenvalues $+1$, $0$, $-1$ that is

$$\frac{1}{2} \left( \vec{\Sigma} \cdot \hat{n} + 1 \right) \bar{\psi}_{RR} = 1$$

$$\frac{1}{2} \left( \vec{\Sigma} \cdot \hat{n} + 1 \right) \bar{\psi}_{RL} = 0$$

$$\frac{1}{2} \left( \vec{\Sigma} \cdot \hat{n} + 1 \right) \bar{\psi}_{LR} = 0$$

$$\frac{1}{2} \left( \vec{\Sigma} \cdot \hat{n} + 1 \right) \bar{\psi}_{LL} = -1$$

In order to illustrate the difficulty with the zero mass limit let us write the expressions for the chiral components in the massive case. By using the decomposition
one can write:

\[ \psi_{R_{a_1} R_{a_2}} = \frac{1}{2} (1 + \gamma^\nu) \sigma^{\mu \nu} c \psi_{a_1 a_2} = \frac{1}{\sqrt{m}} \left( \frac{1}{2} (1 + \gamma^\nu) \sigma^{\mu \nu} c \right)_{a_1 a_2} B_\mu \]

\[ \psi_{L_{a_1} L_{a_2}} = \frac{1}{2} (1 - \gamma^\nu) \sigma^{\mu \nu} c \psi_{a_1 a_2} = \frac{1}{\sqrt{m}} \left( \frac{1}{2} (1 - \gamma^\nu) \sigma^{\mu \nu} c \right)_{a_1 a_2} B_\mu \]  

\[ \psi_{L_{a_1} R_{a_2}} = \frac{1}{2} (1 - \gamma^\nu) c \sigma^{\mu \nu} \psi_{a_1 a_2} = \frac{1}{\sqrt{m}} \left( \frac{1}{2} (1 - \gamma^\nu) \sigma^{\mu \nu} c \right)_{a_1 a_2} B_\mu \]

From (IV.10) one can see that the zero mass limit leads to chiral components that are zero or ill defined. However, from the requirement that \( \psi \) is a symmetric field in its spin indices one can write a decomposition analogous to (II.2) for \( \psi \) that is:

\[ \bar{\psi}_{a_1 a_2}(x) = c_1 A_{\mu}(x) (\gamma^\mu C)_{a_1 a_2} - c_2 F_{\mu \nu}(x) (\sigma^{\mu \nu} C)_{a_1 a_2} \]  

\[ \bar{\psi}_{a_1 a_2}(x) = c_1 A_{\mu}(x) (\gamma^\mu C)_{a_1 a_2} - c_2 F_{\mu \nu}(x) (\sigma^{\mu \nu} C)_{a_1 a_2} \]  

\[ i(\gamma^\nu \partial_\nu A_{\mu} + c_1 A_{\mu}(x) (\gamma^\mu C)_{a_1 a_2} - c_2 F_{\mu \nu}(x) (\sigma^{\mu \nu} C)_{a_1 a_2} = 0 \]  

Multiplying (IV.12) by \((C^{-1})_{a_1 a_2}\) we obtain

\[ \partial_\mu A_{\mu} = 0 \]  

that is, the field \( A_{\mu} \) satisfies the Lorentz condition. Furthermore multiplying (IV.12) by \((C^{-1})_{a_1 a_2}\) we obtain

\[ \partial_\mu F_{\mu \nu} = 0 \]  

Writing \( F_{\mu \nu} \) as

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

we will then, get

\[ \partial_\lambda \partial^\lambda A_\beta - \partial_\beta \partial^\beta A_\lambda = 0 \implies \partial A_\beta = 0 \]

For a massless field we must have following conditions:

\[ \partial_\mu A_\mu = 0 \]

\[ \partial_\mu F_{\mu \nu} = 0 \]

\[ \partial A_\mu = 0 \]

and we note that in this case \( A_{\mu} \) and \( F_{\mu \nu} \) are, in principle independent fields.
The most general form for $F_{\mu\nu}$ is

$$F_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} \epsilon^{\alpha\beta}.$$  

and

$$\hat{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \epsilon^{\alpha\beta}.$$  

By using the decomposition (IV.11) we get the following expressions for

$$\tilde{\psi}_{RR}, \tilde{\psi}_{RL}, \tilde{\psi}_{LR}, \tilde{\psi}_{LL}$$

$$\tilde{\psi}_{R_1 R_2} = \frac{1}{2} (1 + \gamma^5) \alpha_{\gamma_1} \frac{1}{2} (1 + \gamma^5) \beta_{\gamma_2} \psi_{R_1 R_2} = -c_2 \left[\frac{1}{2} (1 + \gamma^5) \sigma^{\mu\nu} C\right]_{\alpha_{\gamma_1} \beta_{\gamma_2}} F_{\mu\nu}$$

$$\tilde{\psi}_{R_1 L_2} = \frac{1}{2} (1 + \gamma^5) \alpha_{\gamma_1} \frac{1}{2} (1 - \gamma^5) \beta_{\gamma_2} \psi_{R_1 L_2} = c_1 \left[\frac{1}{2} (1 + \gamma^5) \sigma^{\mu\nu} C\right]_{\alpha_{\gamma_1} \beta_{\gamma_2}} A_{\mu}$$

$$\tilde{\psi}_{L_1 L_2} = \frac{1}{2} (1 - \gamma^5) \alpha_{\gamma_1} \frac{1}{2} (1 + \gamma^5) \beta_{\gamma_2} \psi_{L_1 L_2} = c_1 \left[\frac{1}{2} (1 + \gamma^5) \sigma^{\mu\nu} C\right]_{\alpha_{\gamma_1} \beta_{\gamma_2}} A_{\mu}$$

$$\tilde{\psi}_{L_1 R_2} = \frac{1}{2} (1 - \gamma^5) \alpha_{\gamma_1} \frac{1}{2} (1 - \gamma^5) \beta_{\gamma_2} \psi_{L_1 R_2} = -c_2 \left[\frac{1}{2} (1 + \gamma^5) \sigma^{\mu\nu} C\right]_{\alpha_{\gamma_1} \beta_{\gamma_2}} F_{\mu\nu}.$$  

As a final result we will show that the right–right and the left–left components, or, equivalently Weyls components give rise to Maxwell’s equations.

For this purpose we will use notation (II.11) which gives the expression of

$$\tilde{\psi}_{RR}, \tilde{\psi}_{LL}$$

in terms of two components spinors.

One has

$$\tilde{\psi}_{RR} = \begin{pmatrix} \varphi \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\psi}_{LL} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varphi^{(2)} \end{pmatrix}.$$  

The symmetry of $\varphi$ allows us to write the following decomposition

$$\varphi(x) = \tilde{\varphi}(x) \cdot \gamma C_4$$

from which we get the following equations for the vector $\tilde{\varphi}(x)$

$$\begin{cases} \tilde{\nabla} \cdot \tilde{\varphi}(x) = 0 \\ \partial \tilde{\varphi}(x) = \tilde{\varphi}(x) \wedge \tilde{\varphi}(x). \end{cases}$$

Now if we define $\varphi^{\mu\nu} = F^{\mu\nu} - \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$

where

$$\varphi^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}.$$  

$A^{\mu}$ being the vector potential and $\varphi^{\mu\nu}$ obeying $\partial \varphi^{\mu\nu} = \partial \varphi^{\mu\nu} F_{\mu\nu} = 0$ and if

$$\mu_{1}(x) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \varphi^{\rho\sigma}(x) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \varphi^{\rho\sigma}(x)$$

$\mu, \alpha, \beta = 1, 2, 3$ equations (IV.18) for $\mu_{1}(x)$ imply Maxwell’s equations that is, equations (IV.18) imply:

$$\begin{cases} \tilde{\nabla} \cdot \tilde{\varphi}(x) = 0 \\ \tilde{\varphi}(x) \wedge \tilde{\varphi}(x) + \partial \tilde{\varphi}(x) = 0 \\ \tilde{\varphi}(x) \wedge \tilde{\varphi}(x) - \partial \tilde{\varphi}(x) = 0 \end{cases}$$

where

$$\begin{cases} \tilde{\varepsilon} = -\partial_{\gamma}\tilde{\varphi} - \tilde{\varphi} \cdot \tilde{\varphi} F_{\mu\nu} \\ \tilde{\eta} = \tilde{\varphi} \wedge \tilde{\varphi} F_{\mu\nu} \end{cases}$$

and
V — INTERACTING FIELDS — QED

Let us consider now the interaction of massless spin 1 fields. We will be mainly concerned with the interaction of these particles with ordinary matter. That is we will be concerned with the most general interaction lagrangian describing massless spin 1 particles interacting with spin $1/2$ massive ($m_1$) particles.

By imposing that the lagrangian is linear in the $\varphi$ fields, then the most general form that is bilinear in the matter field $\eta$ that we can construct with the four independent fields $\psi_{RR}$, $\psi_{RL}$, $\psi_{LR}$, and $\psi_{LL}$ is:

\[ \mathcal{L}_{\text{int}} = A \tilde{\psi}_{RR} \psi_{RR} \eta \eta + B \tilde{\psi}_{RL} \psi_{RL} \eta \eta + D \tilde{\psi}_{LR} \psi_{LR} \eta \eta + E \tilde{\psi}_{LL} \psi_{LL} \eta \eta + \text{h.c.} \]  

(V.1)

where $A$, $B$, $D$, $E$, $F$, $J$, $K$, $L$ are arbitrary constants.

The reason for so many terms is that we shall assume, to start with, that ordinary matter couples with different coupling to the chiral components of the field $\psi$. As a matter of fact we shall see that, in this case, only the zero helicity components couple with ordinary matter.

The number of coupling constants can be reduced to four by imposing Lorentz invariance. In fact using the transformation rules (IV.6) of the fields $\psi_{RR}$, $\psi_{RL}$, $\psi_{LR}$ and $\psi_{LL}$ one can see that, under Lorentz transformations, one has:

\[ \tilde{\psi}_{RR} \eta \eta (\tilde{\psi}_{RR} \eta \eta) \quad \text{if} \quad \det \ell = 1 \]
\[ \tilde{\psi}_{LL} \eta \eta (\tilde{\psi}_{LL} \eta \eta) \quad \text{if} \quad \det \ell = -1 \]

(V.2)

\[ \tilde{\psi}_{RL} \eta \eta (\tilde{\psi}_{RL} \eta \eta) \quad \text{if} \quad \det \ell = 1 \]
\[ \tilde{\psi}_{LR} \eta \eta (\tilde{\psi}_{LR} \eta \eta) \quad \text{if} \quad \det \ell = -1 \]

So that in order to insure Lorentz invariance one has to require that

\[ A = B \]
\[ D = E \]
\[ F = J \]
\[ K = L \]

The most general interaction lagrangian is then:

\[ \mathcal{L}_{\text{int}} = A(\tilde{\psi}_{RR} \eta \eta + \tilde{\psi}_{LL} \eta \eta) + D(\tilde{\psi}_{RL} \eta \eta + \tilde{\psi}_{LR} \eta \eta) + \]
\[ + F(\tilde{\psi}_{RR} \eta \eta + \tilde{\psi}_{LL} \eta \eta) + K(\tilde{\psi}_{RL} \eta \eta + \tilde{\psi}_{LR} \eta \eta) \]  

(V.3)

We now consider the particular lagrangian of interaction obtained for $A = D = K = 0$

\[ \mathcal{L}_{\text{int}} = F(\tilde{\psi}_{RR} \eta \eta + \tilde{\psi}_{LL} \eta \eta) \]  

(V.4)

The total lagrangian is then the following.
\[ \mathcal{L} = \bar{\psi}_{RL}(i\partial \cdot \gamma) \psi_{RR} + \bar{\psi}_{LR}(i\partial \cdot \gamma) \psi_{LL} + \bar{\psi}_{RR}(i\partial \cdot \gamma) \psi_{RL} + \bar{\psi}_{LL}(i\partial \cdot \gamma) \psi_{LR} + \]
\[+ F(\bar{\psi}_{RR} \psi_{LL} \bar{\eta} + \bar{\psi}_{LL} \psi_{RR} \bar{\eta}) + \bar{\eta}(i \gamma - m) \eta. \] 

Writing the Lagrange equations explicitly one has

\[ i \partial \cdot \gamma \bar{\psi}_{LR} + F \bar{\eta} \eta = 0 \]  \hspace{1cm}  \text{(V.6a)}

\[ i \partial \cdot \gamma \bar{\psi}_{RL} + F \bar{\eta} \eta = 0 \]  \hspace{1cm}  \text{(V.6b)}

\[ i \partial \cdot \gamma \bar{\psi}_{RR} = 0 \]  \hspace{1cm}  \text{(V.6c)}

\[ i \partial \cdot \gamma \bar{\psi}_{LL} = 0. \]  \hspace{1cm}  \text{(V.6d)}

Eqs. (V.6a) and (V.6b) impose in particular the following restriction

\[ 2i \epsilon_{\mu} \partial^{\mu} A_{\mu} = F \bar{\eta} \eta \]

\[ i \partial^{\mu} F_{\mu\nu} = 0. \]

In this case we do not get QED.

QED

Let us consider now the following interaction lagrangian for the spin 1 massless particles and the usual matter

\[ \mathcal{L}_{int} = K \left[ \bar{\psi}_{RL} \eta + \bar{\psi}_{LR} \psi \eta \right]. \]  \hspace{1cm}  \text{(V.7)}

By adding the free field lagrangians we end up with the following total lagrangian

\[ \mathcal{L} = \bar{\psi}_{RL}(i\partial \cdot \gamma) \psi_{RR} + \bar{\psi}_{LR}(i\partial \cdot \gamma) \psi_{LL} + \bar{\psi}_{RR}(i\partial \cdot \gamma) \psi_{RL} + \bar{\psi}_{LL}(i\partial \cdot \gamma) \psi_{LR} + \]
\[+ K \left( \bar{\psi}_{RL} \eta + \bar{\psi}_{LR} \psi \eta \right) + \bar{\eta}(i \gamma - m) \eta. \]  \hspace{1cm}  \text{(V.8)}

One of the most interesting aspects of this approach is that although only some chiral components couple with the usual matter, all chiral components should be considered as dynamical variables. That is one should write five Euler–Lagrange equations:

\[ \frac{\partial \mathcal{L}}{\partial \bar{\psi}_{RR}} - \partial_{\mu} \left[ \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \bar{\psi}_{RL})} \right] = 0 \]

\[ \frac{\partial \mathcal{L}}{\partial \bar{\psi}_{RL}} - \partial_{\mu} \left[ \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \bar{\psi}_{LL})} \right] = 0 \]

\[ \frac{\partial \mathcal{L}}{\partial \bar{\psi}_{LL}} - \partial_{\mu} \left[ \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \bar{\psi}_{LR})} \right] = 0 \]

\[ \frac{\partial \mathcal{L}}{\partial \bar{\eta}} - \partial_{\mu} \left[ \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \eta)} \right] = 0. \]
Equations (V.9) and \( \mathcal{L} \) given by (V.8) leads to:

\[
\begin{align*}
\imath \omega \partial \phi \tilde{\phi}_{RL} &= 0 \quad \text{(V.10a)} \\
\imath \omega \partial \phi \tilde{\phi}_{RR} + K \eta \partial \eta &= 0 \quad \text{(V.10b)} \\
\imath \omega \partial \phi \tilde{\phi}_{LL} + K \eta \partial \eta &= 0 \quad \text{(V.10c)} \\
\imath \omega \partial \phi \tilde{\phi}_{LR} &= 0 \quad \text{(V.10d)} \\
2K \eta \left\{ \tilde{\phi}^\dagger_{RL} \phi_{LR} + \tilde{\phi}^\dagger_{LR} \phi_{RL} \right\} - m \eta - i \partial \mu (\eta) \gamma^\mu &= 0 \quad \text{(V.10e)}
\end{align*}
\]

From Eq. (V.10e) it follows that:

\[
2K \eta \left\{ \tilde{\phi}^\dagger_{RL} \phi_{LR} + \tilde{\phi}^\dagger_{LR} \phi_{RL} \right\} \eta_{a_1 a_2} - m \eta_{a_1 a_2} - i \partial \mu (\eta) \gamma^\mu \eta_{a_1 a_2} = 0
\]

Or replacing \( \tilde{\phi}_{RL} \) and \( \tilde{\phi}_{LR} \) by expression (IV.16)

\[
-2K \eta c_{a_1} A^{(a_1)}_{a_2} A_{a_2} \eta_{a_1 a_2} - m \eta_{a_1 a_2} - i \partial \mu (\eta) \gamma^\mu \eta_{a_1 a_2} = 0 \quad \text{(V.11)}
\]

Taking the hermitian conjugate of (V.11) one gets

\[
(i \omega - m) \eta_{a_1 a_2} \eta_{a_2} = 2K \eta c_{(a_1)} A_{a_2} \eta_{a_2} \quad \text{(V.12)}
\]

Note that by imposing \( 2K c_1 = e \) we get the usual minimal coupling.

Now adding (V.10b) and (V.10c) and replacing \( \tilde{\phi}_{RR} \) and \( \tilde{\phi}_{LL} \) by there expressions

(IV.16) we get

\[
-i c_2 (\gamma^{\alpha \mu} \mu C) A_{a_1} \eta_{a_2} \partial \alpha \gamma^\mu + 2K \eta c_{a_1} \eta_{a_2} = 0 \quad \text{(V.13)}
\]

Multiplying (V.13) by \( (C^{-1})^{\alpha \beta}_{a_1 a_2} \) we get:

\[
\partial \mu (\gamma^\alpha \gamma^\beta) \eta_{a_1 a_2} = \frac{K}{4 \eta_{a_1 a_2} \gamma^\beta \eta} \quad \text{(V.14)}
\]

We see that equations (V.14) can be obtained from equations (III.18).

Similarly adding (V.10a) and (V.10d) and replacing \( \tilde{\phi}_{LR} \) and \( \tilde{\phi}_{RL} \) by expressions

(IV.16) we get:

\[
i c_1 (\gamma^{\alpha \mu} \mu C) A_{a_1} \eta_{a_2} \partial \alpha A^\mu = 0 \quad \text{(V.15)}
\]

So multiplying (V.15) by \( (C^{-1})^{a_1}_{a_2} \) we have:

\[
\partial^\mu A^\mu = 0 \quad \text{(V.16)}
\]

that is, one gets Lorentz condition.

We have seen in IV that the most general form \( F_{\mu \nu} \) is

\[
F_{\mu \nu} = f_{\mu \nu} + i f_{\mu \nu} \quad \text{(V.17)}
\]

Replacing \( F_{\mu \nu} \) by (V.17) in Eq. (V.14) we get an alternative form for Eq. (V.14) that is
\[ \begin{align*}
\partial_\mu \tilde{\eta} & = \frac{K}{4c_2} \tilde{\gamma}^\mu \eta \\
\partial_\mu \tilde{\rho}_\mu & = 0 \quad \text{(V.18)}
\end{align*} \]

These are Maxwell's equations in Lorentz gauge if one writes

\[ f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

and imposes the restriction

\[ \frac{K}{4c_2} = e \]

In order to see this equivalence in terms of the observables \( \tilde{E} \) and \( \tilde{H} \) one writes as usual

\[ f^{sk} = E^k \]

\[ d^{sk} = \epsilon^{jlk} H^l \]

Equations (V.18) give rise to the following equations

\[ \tilde{\nabla} \cdot \tilde{E} = e \tilde{\gamma}^\rho \tilde{\eta} \]

\[ \tilde{\nabla} \cdot \tilde{H} = -e \tilde{\eta} \tilde{\gamma} \tilde{\eta} \]

\[ \tilde{\nabla} \cdot \tilde{H} = 0 \]

\[ \tilde{\nabla} \cdot \tilde{E} + \tilde{\nabla} \wedge \tilde{H} = 0 \quad \text{(V.19)} \]

(V.19) are exactly Maxwell's equations.

The total lagrangian as a function of \( A_\mu \) and \( F_{\mu\nu} \)

Replacing \( \tilde{\psi}_{RR}, \tilde{\psi}_{RL}, \tilde{\psi}_{LR} \) and \( \tilde{\psi}_{LL} \) by the expressions (IV.16) we obtain for the free spin 1 massless lagrangian

\[ L = \tilde{\psi}_{RL} (i \tilde{\nabla} \otimes 1) \tilde{\psi}_{RR} + \tilde{\psi}_{LR} (i \tilde{\nabla} \otimes 1) \tilde{\psi}_{RL} + \tilde{\psi}_{RR} (i \tilde{\nabla} \otimes 1) \tilde{\psi}_{LR} + \tilde{\psi}_{LL} (i \tilde{\nabla} \otimes 1) \tilde{\psi}_{LR} = \]

\[ = 4c_1c_2 F^{\mu\nu} F_{\mu\nu} - Kc_1 A_\mu^* \tilde{\gamma}^\mu \tilde{\eta} + \tilde{\eta} (i \tilde{\nabla} - m_i) \tilde{\eta} \quad \text{(V.21)} \]

Note that the product \( c_1c_2 \) is dimensionless and we impose now that

\[ c_1c_2 = \frac{1}{4} \]

\( A_\mu \) and \( F_{\mu\nu} \) as independent fields

Replacing \( \tilde{\psi}_{RR}, \tilde{\psi}_{RL}, \tilde{\psi}_{LR} \) and \( \tilde{\psi}_{LL} \) by the expressions (IV.16) we obtain for the free spin 1 massless lagrangian (IV.4)

\[ L_0 = 4c_1c_2 \left( F^{\mu\nu} \left[ \tilde{\partial} \tilde{A}^\mu - \tilde{\partial} \tilde{A}^\nu \right] - A_\mu^* \left[ \tilde{\partial} \tilde{F}_{\nu\mu} - \tilde{\partial} \tilde{F}_{\mu\nu} \right] \right) \]

where we have considered \( A^\mu \) and \( F^{\mu\nu} \) as independent fields.

The total lagrangian (V.8) may be written

\[ L = 4c_1c_2 F^{\mu\nu} \left[ \tilde{\partial} \tilde{A}^\mu - \tilde{\partial} \tilde{A}^\nu \right] + 4c_1c_2 A_\mu^* \left[ \tilde{\partial} \tilde{F}_{\nu\mu} - \tilde{\partial} \tilde{F}_{\mu\nu} \right] - 

\[ - Kc_1 A_\mu^* \tilde{\gamma}^\mu \tilde{\eta} + \tilde{\eta} (i \tilde{\nabla} - m_i) \tilde{\eta} \quad \text{(V.22)} \]
Writing Lagrange's equations for $A^\mu$, $F^{\mu\nu}$ and $\eta$ we obtain

$$\frac{\partial L}{\partial A^\nu^*} - \partial^\lambda \left( \frac{\partial L}{\partial A^\lambda A^\nu^*} \right) = 0 \implies \partial^\nu F^\mu_\nu = \frac{K}{8c_2} \eta \gamma^\mu \eta \tag{V.23}$$

$$\frac{\partial L}{\partial \eta} - \partial^\mu \left( \frac{\partial L}{\partial (\partial^\mu \eta)} \right) = 0 \implies (i\partial - m_\eta) \eta = Kc_1 A^+ \eta. \tag{V.24}$$

**VI - QUANTIZATION**

In this chapter we will consider the quantization of massive fields within the BW theory. We propose also an extension to the zero mass case. We show that there is no basic distinction between this and the usual approach. The only interesting point is that it is possible to quantize the BW fields.

We quantize the BW fields by imposing the following commutation rules for BW's massive fields:

$$\left[ \psi_{a_1 a_2 \ldots a_{2\nu}} (x), \psi_{b_1 b_2 \ldots b_{2\nu}} (y) \right] =$$

$$= (-1)^{s^2_{\nu} - \kappa} \sum_\mathcal{P} \frac{(iL^0 + m)^{a_1 b_1}}{(iL^0 + m)^{2\nu} a_2 b_2} \Delta (x-y) \tag{VI.1}$$

where $a_1, \ldots, a_{2\nu}$, $b_1, \ldots, b_{2\nu}$, $\Delta (x-y)$ is the Jordan Pauli function, $\kappa$ is a constant to be determined, $\mathcal{P}$ denotes all possible permutations among the spinor indices and where we use below convention

$$[\psi, \bar{\psi}]_a = \psi \bar{\psi} + (-1)^{s^2_{\nu}} \bar{\psi} \psi \tag{VI.2}$$

$s$ being the spin of considered field.

One writes the BW fields as a linear combination of the symmetric matrices of spinor-space and introduce so new fields which are the coefficients of this expansion. Imposing commutation rules (VI.1) and replacing the BW fields by the precedent expansion, we obtain commutation rules for these new fields.

For a massive spin 1 particle we adopt the decomposition of the BW field in spinor space:

$$\psi_{a_1 a_2} (x) = \sqrt{m} \left( B_\mu (x) (\gamma^\mu C)_{a_1 a_2} - \frac{i}{2m} G_{\mu\nu} (x) (\gamma^\mu \gamma^\nu C)_{a_1 a_2} \right).$$
By writing the plane wave expansion of the new field $B_\mu$ as:

$$B_\mu(x) = \int \frac{d^3p}{(2\pi)^3/2} \sum_n \frac{1}{2\pi} \left\{ A(p,\lambda) \epsilon^\mu(p,\lambda) e^{i p x} + B^{+\mu}(p,\lambda) \epsilon^{+\mu}(p,\lambda) e^{i p x} \right\}$$

(VI.3)

the properties of the polarization $\epsilon^\mu$ (deduced from the property (III.8) of the $B^\mu$ field) are:

$$p^\mu \epsilon_\mu(p,\lambda) = 0 .$$

(VI.4)

Adopting the normalization

$$\epsilon^{+\mu}(p,\lambda) \epsilon_\mu(p,\lambda') = -\delta_{\lambda\lambda'} ,$$

(VI.5)

we obtain the sum over the polarizations

$$X^{\mu\nu}(p) = \sum_\lambda \epsilon^\mu(p,\lambda) \epsilon^{+\nu}(p,\lambda) = -\left[ g^{\mu\nu} - p^\mu p^\nu / m^2 \right] .$$

(VI.6)

Finally we can deduce the Feynman propagator for the $\psi$ field

$$i D_\mu^\nu(p) = \frac{1}{-p^2 - m^2 + i\epsilon} X^{\mu\nu}(p) .$$

(VI.7)

Massless BW fields$^5$)

As shown in chapter II, for a massless field the fundamental things are the two components spinors of rank $2s, \varphi, \varphi', \ldots \varphi^{(2s-1)}$.  

For the massless case we quantize the two component Weyl spinor $\varphi$ (II.11) assuming commutation rules (VI.8),

$$\left[ \varphi_{b_1 \ldots b_{2s}}(x), \varphi^{+\dagger}_{b_1 \ldots b_{2s}}(y) \right] =$$

$$= (-i)^{2s+1} \sum_{\mathcal{P}} \prod_{\lambda=1}^{2s} i \left( \partial_0 - i \mathbf{\sigma} \cdot \mathbf{p} \right) \mathbf{\sigma}_{b_{2s+1}} \ldots i \left( \partial_0 - i \mathbf{\sigma} \cdot \mathbf{p} \right) \mathbf{\sigma}_{b_1} \mathbf{D}(x-y)$$

(VI.8)

where $\mathbf{D}(x-y)$ is the Jordan Pauli non massive function, $\kappa$ is a constant to be determined, $\mathcal{P}$ denotes all possible permutations among the spinor indices and where we use convention (VI.7).

Then we expand the two component $\varphi$ fields in term of the new field $\tilde{f}$ (IV.17); imposing (VI.8) we obtain commutation relations for the $\tilde{f}$ fields.

In the particular case of a spin 1 particle where (IV.19)

$$f_\mu(x) = \frac{1}{2} \epsilon_\mu\alpha\beta \varphi^{\alpha\beta}(x)$$

postulating the plane wave expansion of $A^\mu(x)$

$$A^\mu(x) = \frac{1}{(2\pi)^3/2} \int \frac{d^3p}{2\pi} \sum_{\lambda=1}^{2s} \epsilon^\mu(p,\lambda) \left\{ A(p,\lambda) e^{i p x} + A^{+\mu}(p,\lambda) e^{i p x} \right\}$$

(VI.9)

where $p = (p^0, \mathbf{p})$

$$[A(p,\lambda), A^{+\mu}(p',\lambda')] = \delta(\mathbf{p} - \mathbf{p}') \delta_{\lambda\lambda'}$$

$\lambda$ and $\lambda'$ are polarizations and using (IV.19) we obtain one more expression for the commutator between $f$ fields.
We obtain the following expression for the sum over polarization for a massless particle of spin 1, momentum $p$, massless:

$$\tilde{X}^{\mu \nu} = \sum_{i=1}^{2} \tilde{v}^{\mu}(p_\lambda) \tilde{z}^{\nu \lambda}(p_\lambda) = -\eta^{\mu \nu} \eta^\alpha \eta^\beta \eta^\gamma$$  \hspace{1cm} \text{(VI.10)}

where we have adopted Bjorken's notation.

$$\eta^\mu = (1,0,0,0)$$

$$\tilde{\eta}^\mu = \frac{\sqrt{(p \cdot \eta)^2 - p^2}}{\eta^\mu - (p \cdot \eta) \eta^\mu}$$

Feynman rules:

We take $L_{\text{int}}$ (III.13) with $g_s = 0$ as a starting point:

$$L_{\text{int}} = g_s \bar{\psi}_{a_1 a_2} \eta_{a_3 a_4} \eta_{a_5 a_6} = g_s \bar{\psi} \gamma \eta \eta'$$

We assume that $\psi$ corresponds to a particle of four-momentum $p_1$ and spin 1, $\eta$ has four-momentum $p_2$ and $\eta'$ has four-momentum $p_3$ and spin $\frac{1}{2}$.

So we obtain the following result:

$$-i \sqrt{m} \left\{ (C^{-\mu \nu})_{a_3 a_4} \eta_{a_5 a_6} \eta_{a_7 a_8} \right\}$$ \hspace{1cm} \text{(VI.11)}

Consider now the 3 below cases:

1) $\eta$ and $\eta'$ are positive energy particles, so

$$\bar{\eta} = \bar{\gamma} = -\bar{\psi} C$$

the vertex above takes the form

$$i \sqrt{m} \bar{\psi}(p_2) \left\{ \eta^{\mu} - \frac{1}{m} (p_1)_\nu \sigma^{\mu \nu} \right\} \psi(p_3)$$

2) $\eta$ and $\eta'$ are negative energy particles, so

$$\bar{\eta} = \bar{\gamma} = -\bar{\psi} C$$

the vertex takes the form

$$i \sqrt{m} \bar{\psi}(p_2) \left\{ \eta^{\mu} - \frac{1}{m} (p_1)_\nu \sigma^{\mu \nu} \right\} \psi(p_3)$$

3) $\eta$ is a positive energy particle and $\eta'$ is a negative energy particle.

$$\bar{\eta} = \bar{\gamma} = -\bar{\psi} C$$

the vertex takes the form

$$i \sqrt{m} \bar{\psi}(p_2) \left\{ \eta^{\mu} - \frac{1}{m} (p_1)_\nu \sigma^{\mu \nu} \right\} \psi(p_3)$$
Appendix.

- The metric used is $g_{\mu\nu} = (1,-1,-1,-1)$

- Dirac's matrices commutation rules

\[
\frac{i}{2} [\gamma^\mu, \gamma^\nu] = \varepsilon^{\mu\nu}
\]

\[
\gamma^\mu = \frac{i}{2} [\gamma^\rho, \gamma^\rho] ; \ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

$\gamma^0 = (\gamma^0, 0)$

$\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$

$\bar{\psi} = \gamma^0 \psi$

$\Sigma = \bar{\psi} \gamma^\mu \gamma^\nu \psi$

- Pauli matrices are defined below by

\[
\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

By convention $\sigma^\mu = (\sigma^\mu = 1, \beta)$ designate $(\sigma^\mu)_{\alpha\beta}$.

We have used the property that:

\[
(\sigma^2)_{\alpha\beta} = -(\sigma^2)_{\alpha\beta}
\]

$C_1$ denotes the charge conjugation matrix in (2,2) space and obeys

$C_1 = -C_{\beta}C_1^\dagger = C_1^\dagger$.

- In chiral representation

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}
\]

$\gamma^\mu_{\alpha\beta} = \begin{bmatrix} 0 & -(\sigma^\mu)_{\alpha\beta} \\ -(\sigma^\mu)_{\beta\alpha} & 0 \end{bmatrix}$

- Charge conjugation matrix $(C)_{\alpha\beta} = \begin{bmatrix} -i\sigma^3_{\alpha\beta} = C_1 & 0 \\ 0 & i\sigma^3_{\beta\alpha} = C_1 \end{bmatrix}$
VI - CONCLUSIONS

In this paper we have extended, to massless particles of arbitrary spin, the usual spin 1/2 chiral components. The extended chiral components are eigenstates of generalized helicity operators. In a particle of spin s the eigenvalues of the helicity states are, as expected, in the range [−s, s]. The chiral components associated to the −s and +s eigenvalues satisfy Weyl’s equations. The other components satisfy equations, in terms of two components spinors, analogous to Weyl’s.

We have made a systematic analysis of the spin 1 particles. In this case there are four chiral components: \( \psi_{RR}, \psi_{LL}, \psi_{RL}, \) and \( \psi_{LR} \). The helicity eigenvalue +1 (−1) is associated to \( \psi_{RR}, \psi_{LL} \) whereas the 0 eigenvalue is associated to the components \( \psi_{RL}, \psi_{LR} \). These components, on the other hand, have a very simple meaning in this case. They are associated to observable fields (like the electric and magnetic fields) or to potentials. That is:

\[
\begin{align*}
\psi_{RR} \quad &\psi_{LL} \quad \text{observable fields (E or H)} \\
&\quad \text{(Gauge invariant)}
\end{align*}
\]

\[
\begin{align*}
\psi_{LR} \quad &\psi_{RL} \quad \text{potentials (A_\mu)} \\
&\quad \text{(not Gauge invariant)}
\end{align*}
\]

We have shown that all chiral components are, in fact, important dynamical variables for getting a complete description of electrodynamics. The equations associated to some components are just Maxwell’s equations whereas other equations give constraints in the potentials, that, in this case, is Lorentz condition.

One of the advantages of this approach is that it allows to formulate electrodynamics in terms of potentials, by using two components spinors, or in terms of the observable fields (Maxwell equation) E and H.

The next question that we asked ourselves is if ordinary matter prefers, or not, to couple with some chiral components. If matter does not distinguish between chiral components, then it should couple with the field \( \psi \). That is not the case.

On the other hand, if ordinary matter couples only with the \( \psi_{RR}, \psi_{LL} \) we would have electrodynamics formulated entirely in terms of observables. That is not the alternative that nature chooses either.

The usual QED is compatible only with a theory in which only \( \psi_{LR}, \psi_{RL} \) couples with the ordinary matter. From this point of view QED is another example of an asymmetric interaction between chiral components.

The coupling of ordinary matter with some chiral components, in the case of the photon, does not have any consequence like parity violation. It is just a dynamical consequence that in this case is the minimal coupling.
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