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LEFT-RIGHT ASYMMETRY AND MINIMAL COUPLING

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ABSTRACT

We extend, to arbitrary spin massless particles, the usual chiral components associated to massless spin $\frac{1}{2}$ particles. In the case of spin 1 particles the nonzero generalized helicity components satisfies Weyl's equations and are associated to observables (Electric and Magnetic Fields) whereas the zero generalized helicity components are related to non–observables (Electromagnetic Potentials). We show that nature, as described by QED, is asymetric with regard to left and right since matter couples only to some chiral components of the zero mass spin 1 field. In this way we have shown that the minimal coupling substitution is just a physical consequence of the left–right asymetry of nature.

I – INTRODUCTION

For massless spin $\frac{1}{2}$ particles one can define chiral components ψ_L and ψ_R of a basic field ψ as follows:

$$\psi_R = \frac{1}{2}(I + \gamma^5)\,\psi$$

$$\psi_L = \frac{1}{2}(I - \gamma^5)\,\psi \quad .$$

It is an intriguing aspect of the fundamental interactions that, at least at low energies, the right and left components interact in a different way with ordinary matter. As a matter of fact there is no evidence at all that the right hand component interacts with ordinary matter.

Nature, at low energies, is definetly asymetric with regard to left and right. The first clue in this direction was given by the old V–A theory of the weak interactions[1)]. The empirical evidence for this asymetry comes from the breakdown of space reflection invariance by the weak interaction[2)].

This paper deals with the question on whether there are other evidences in nature of an asymetric coupling of chiral components of zero mass particles to ordinary matter. Besides the neutrinos the next, nontrivial, zero mass particles that couples to ordinary particles are the photons. The question that we adress ourselves in this paper is whether the coupling of photons reflects some kind of asymetry between the coupling of chiral components to ordinary matter. We find that this is indeed the case that minimal coupling is nothing but a reflex of this asymetry.

Obviously one needs to specify what we mean by chiral components of a spin zero field. These components have a well defined meaning if we work within the Bargmann Wigner (B.W.) method. If ψ represents the B.W. field associated to a spin 1 particle than one can define generalized chiral components for the field as follows:

$$\psi_{RR} = \frac{1}{2}(I + \gamma^5) \otimes \frac{1}{2}(I + \gamma^5)\, \psi$$

$$\psi_{LL} = \frac{1}{2}(I - \gamma^5) \otimes \frac{1}{2}(I - \gamma^5)\, \psi$$

$$\psi_{RL} = \frac{1}{2}(I + \gamma^5) \otimes \frac{1}{2}(I - \gamma^5)\, \psi$$

$$\psi_{LR} = \frac{1}{2}(I - \gamma^5) \otimes \frac{1}{2}(I + \gamma^5)\, \psi$$

We will show that QED can be completly formulated by taking only the interaction of ψ_{LR} and ψ_{RL} with ordinary matter fields. Within the context of QED, we will see that the usual minimal substitution prescription $\partial_\mu \to \partial_\mu - ieA_\mu$ is nothing but a reflex of the left–right asymetry in the case of spin 1 particles.

The plan of this paper is the following:

In chapter II we extend to particle of arbitrary spins s, the usual spin $\frac{1}{2}$ chiral components. This extension is possible in the context of B.W. theory. The totally right (left) components have generalized helicity components s (–s) and obey the generalized Weyl's equation.

As an exercise, and in order to illustrate how the B.W. theory works, we present in chapter III the B.W. theory for spin 1 massive particles. The interesting point here is that clearly B.W. theory leads to a complete description of massive particles by associating to these particles a symmetric rank 2 tensor field instead of associating particles to a rank 1 tensor (the usual procedure). The subsidiary condition $\partial^\mu B_\mu = 0$, for instance, follows naturally from the decomposition of the basic B.W. field into the spinor space and the B.W. equation.

In chapter IV we make the extension of the B.W. theory to massless spin 1 particles. In this case it is straightforward to show that the two components of the chiral

components are associated to the potential whereas the other components are associated to the potentials.

In chapter V we formulate QED in terms of the chiral components ψ_{RR}, ψ_{RL}, ψ_{LR} and ψ_{LL}. Here we show explicity that one can formulate QED as long as matter couples only to some components of the chiral fields. That is, QED is manifestly left–right asymetric.

In chapter VI we touch on the question of the quantization of the B.W. fields. This is achieved by imposing appropriate commutation relation among the B.W. components.

We end this paper with a chapter dedicated to conclusions.

II – THE BASIC FRAMEWORK

In this chapter we will describe the general framework for studying massive and massless particles of arbitrary spin.

Our starting point for describing massive particles is Bargmann Wigner (B.W.) theory[3]. In the case of massless particles we define generalized chiral components that are an extension for particles of any spin of the usual ones for spin $\frac{1}{2}$ particles. Some of these components satisfy the usual Weyl's equations, the others satisfy equations analogous to Weyl's equations.

Remembering that a rank one spinor transforms, under Poincaré transformation as

$$\eta(x) \longrightarrow \eta'(x') = D(\ell)\,\eta(x)$$

where, in order to ensure the relativistic invariance of Dirac's equations one has to require

$$D^{-1}(\ell)\,\gamma^\mu D(\ell) = \ell^\mu{}_\nu\,\gamma^\nu$$

then, a $2s$ rank spinor $\psi(x)$ transforms as

$$\psi(x) \longrightarrow \psi'(x') = \underbrace{D(\ell) \otimes \cdots \otimes D(\ell)}_{2s\ \text{factors}}\,\psi(x) \;.$$

Analogously, the $2s$ rank spinor $\bar\psi(x)$ defined by

$$\bar\psi(x) = \psi^+(x)\,\underbrace{\gamma^0 \otimes \cdots \otimes \gamma^0}_{2\ s\ \text{factors}}$$

transforms like

$$\bar\psi(x) \longrightarrow \bar\psi'(x') = \bar\psi(x)\,\underbrace{D^{-1}(\ell) \otimes \cdots \otimes D^{-1}(\ell)}_{2s\ \text{factors}}$$

where we have used the property that

$$\gamma^0 D^+(\ell)\,\gamma^0 = D^{-1}(\ell) \;.$$

Throughout this paper we use the notation of Bjorken and Drell book[4]. In an appendix of this paper we present some of the matrices and notation that we have used.

Within the BW theory[3] a massive particle of mass m and spin s is described by a 2s rank spinor field

$$\psi_{a_1 a_2 \ldots a_{2s}}(x) \qquad a_k = 1, 2, 3, 4$$

that is symmetric in its spin indices, obeying a system of $2s$ Dirac equations:

$$i\,\partial_{a_k a'_k}\,\psi_{a_1\ldots a'_k\ldots a_{2s}}(x) = m\,\psi_{a_1\ldots a_k\ldots a_{2s}}(x)$$
$$k = 1, 2s$$

or in a different notation

$$\begin{aligned}
& i\partial\!\!\!/ \otimes I \otimes \cdots \otimes I\,\psi = m\,\psi \\
& I \otimes i\partial\!\!\!/ \otimes \cdots \otimes I\,\psi = m\,\psi \\
& \vdots \\
& I \otimes I \otimes \cdots \otimes i\partial\!\!\!/\,\psi = m\,\psi \;.
\end{aligned} \tag{II.1}$$

The symmetry in the spin indices of the BW spinor field ψ allows us to express this field in terms of a linear combination of symmetric matrices of the spinor space. In this expansion, new fields appear (which, as illustrated in the case of spin 1 particles, are the usual spin 1 fields) which are coefficients in the expansion. Some properties for these new fields can be deduced from Dirac equation (II.1) for the BW field.

For the sake of completeness let us write the decomposition of the rank 2 spinor field $\psi_{a_1a_2}$ associated to a mass m and spin 1 particle in terms of the symmetric matrices $(\gamma^\mu C)$ and $(\sigma^{\mu\nu}C)$. $\psi_{a_1a_2}$ admits the following decomposition:

$$\psi_{a_1a_2}(x) = \sqrt{m}\left\{ B_\mu(x)(\gamma^\mu C)_{a_1a_2} - \frac{1}{2m} G_{\mu\nu}(x)(\sigma^{\mu\nu}C)_{a_1a_2}\right\} \tag{II.2}$$

where the field $B_\mu(x)$ is a vector field (that as will be shown later is the usual spin 1 field), and $G_{\mu\nu}(x)$ is a field not yet determined and that, in principle should involve derivatives of the $B_\mu(x)$ field. This is actually what happens. As will be shown in the next section, from (II.1) and (II.2) it follows that

$$G_{\mu\nu}(x) = \partial_\nu B_\mu(x) - \partial_\mu B_\nu(x) \quad . \tag{II.3}$$

In the case of massless spin $\frac{1}{2}$ particles, one starts with Weyl's equation. In this case the basic set of equations is:

$$\begin{cases} (-i\sigma^0\partial_0 - i\vec{\sigma}\cdot\vec{\nabla})\xi(x) = 0 \\ (-i\sigma^0\partial_0 + i\vec{\sigma}\cdot\vec{\nabla})\dot{\chi}(x) = 0 \end{cases} \tag{II.4}$$

where $\xi(x)$ and $\dot{\chi}(x)$ are two component spinors. These spinors are eigenstates of the helicity operator $\frac{1}{2}\vec{\sigma}\cdot\vec{n}$ $\left[\vec{n} = \frac{\vec{p}}{|\vec{p}|}\right]$ with eigenvalues $\pm\frac{1}{2}$:

$$\begin{cases} \frac{1}{2}(\vec{\sigma}\cdot\vec{n})\xi = \frac{1}{2}\xi \\ \frac{1}{2}(\vec{\sigma}\cdot\vec{n})\dot{\chi} = -\frac{1}{2}\dot{\chi} \end{cases} \quad . \tag{II.5}$$

If one write a four component spinor as

$$\tilde{\psi} = \begin{bmatrix} \xi \\ \dot{\chi} \end{bmatrix}$$

then one can define, as usual, the right and left components as

$$\begin{cases} \tilde{\psi}_R = \frac{1}{2}(I+\gamma^5)\,\tilde{\psi} \\ \tilde{\psi}_L = \frac{1}{2}(I-\gamma^5)\,\tilde{\psi} \end{cases} \quad . \tag{II.6}$$

With the definition (II.6) one can then show that Weyl's equations (II.4) are equivalent to the equations:

$$\begin{cases} i\partial\!\!\!/\tilde{\psi}_R = 0 \\ i\partial\!\!\!/\tilde{\psi}_L = 0 \end{cases} \quad .$$

For particles of arbitrary spin we assume (like in BW's theory for massive particles) that a particle of spin s, without mass is described by a spinor

$$\tilde{\psi}_{a_1\ldots a_k\ldots a_{2s}}(x) \qquad a_k = 1, 2, 3, 4$$

of rank 2s, symmetric in its spin variables, which obeys a system of 2s Dirac's equations:

$$i\,\partial_{a_k a_k'}\,\tilde{\psi}_{a_1\ldots a_k'\ldots a_{2s}}(x) = 0 \qquad \text{with} \qquad k = 1, 2s \quad . \tag{II.7}$$

The definition of chiral components for massless particles of arbitrary spin is:

$$\tilde{\psi}_{R\ldots R}(x) \equiv \tilde{\psi}_{R_{a_1}\ldots R_{a_{2s}}}(x) = \frac{1}{2}(I+\gamma^5)_{a_1 a_1'}\frac{1}{2}(I+\gamma^5)_{a_2 a_2'}\cdots\frac{1}{2}(I+\gamma^5)_{a_{2s} a_{2s}'}\tilde{\psi}_{a_1'\ldots a_{2s}'}(x)$$

$$\tilde{\psi}_{R\ldots RL}(x) \equiv \tilde{\psi}_{R_{a_1}\ldots R_{a_{2s-1}} L_{a_{2s}}}(x) =$$

$$= \frac{1}{2}(I+\gamma^5)_{a_1 a_1'}\cdots\frac{1}{2}(I+\gamma^5)_{a_{2s-1}\, a_{2s-1}'}\cdot\frac{1}{2}(I-\gamma^5)_{a_{2s} a_{2s}'}\tilde{\psi}_{a_1'\ldots a_{2s}'}(x)$$

$$\tilde{\psi}_{L\ldots L}(x) \equiv \tilde{\psi}_{L_{a_1}\ldots L_{a_{2s}}}(x) = \frac{1}{2}(I-\gamma^5)_{a_1 a_1'}\cdots\frac{1}{2}(I-\gamma^5)_{a_{2s}\, a_{2s}'}\tilde{\psi}_{a_1'\ldots a_{2s}'}(x) \quad . \qquad \text{(II.8)}$$

From the equation (II.7) we may derive the following equations:

$$\begin{aligned}
&i\not{\partial}_{a_1 a_1'}\tilde{\psi}_{R_{a_1'}R_{a_2}\ldots R_{a_{2s}}}(x) = 0\\
&i\not{\partial}_{a_1 a_1'}\tilde{\psi}_{R_{a_1'}R_{a_2}\ldots L_{a_{2s}}}(x) = 0\\
&\vdots\\
&i\not{\partial}_{a_1 a_1'}\tilde{\psi}_{L_{a_1'}L_{a_2}\ldots L_{a_{2s}}}(x) = 0 \quad . \qquad \text{(II.9)}
\end{aligned}$$

It is clear that if $\tilde{\psi}$ transforms under the Poincare group like the tensor product of 2s bispinors[5])

$$\tilde{\psi} \sim \begin{bmatrix}\xi_I\\ \dot{\chi}_I\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix}\xi_{IIs}\\ \dot{\chi}_{IIs}\end{bmatrix}$$

then, in chiral representation, representation that we will use from now on in the massless case, $\tilde{\psi}_{R\ldots R}$, $\tilde{\psi}_{R\ldots RL}$, ..., $\tilde{\psi}_{L\ldots L}$, will transform like:

$$\tilde{\psi}_{R\ldots R} \sim \begin{bmatrix}\xi_I\\ 0\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix}\xi_{IIs}\\ 0\end{bmatrix} = \begin{bmatrix}\xi_I \cdots \xi_{IIs}\\ 0\\ \vdots\\ 0\end{bmatrix}$$

$$\tilde{\psi}_{R\ldots RL} \sim \begin{bmatrix}\xi_I\\ 0\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix}\xi_{IIs-1}\\ 0\end{bmatrix}\otimes\begin{bmatrix}0\\ \dot{\chi}_{II_s}\end{bmatrix} = \begin{bmatrix}0\\ \xi_I \cdots \xi_{IIs-1}\dot{\chi}_{IIs}\\ 0\\ \vdots\\ 0\end{bmatrix}$$

$$\vdots$$

$$\tilde{\psi}_{L\ldots L} \sim \begin{bmatrix}0\\ \dot{\chi}_I\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix}0\\ \dot{\chi}_{II_s}\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\\ \dot{\chi}_I \cdots \dot{\chi}_{IIs}\end{bmatrix} \quad . \qquad \text{(II.10)}$$

If we adopt the following notation

$$\tilde{\psi}_{R\ldots R} \sim \begin{bmatrix}\varphi\\ 0\\ \vdots\\ 0\end{bmatrix}$$

$$\tilde{\psi}_{R\ldots RL} \sim \begin{bmatrix}0\\ \varphi'\\ 0\\ \vdots\\ 0\end{bmatrix}$$

$$\vdots$$

$$\tilde{\psi}_{L\ldots L} \sim \begin{bmatrix}0\\ \vdots\\ 0\\ \varphi^{(2s-1)}\end{bmatrix} \quad . \qquad \text{(II.11)}$$

Where φ, φ', $\cdots$, $\varphi^{(2s-1)}$, are respectively the only non zero components of $\tilde{\psi}_{R\ldots R}$, $\tilde{\psi}_{R\ldots RL}$, $\cdots$, $\tilde{\psi}_{L\ldots L}$ and are two component spinors of rank 2s:

$$\varphi = \varphi_{b_1 b_2 \cdots b_{2s}}$$

$$\varphi' = \varphi'_{b_1 b_2 \cdots b_{2s}} \qquad\qquad b_1, b_2, \ldots b_{2s-1} = 1, 2$$

$$\vdots$$

$$\varphi^{(2s-1)} = \varphi^{(2s-1)}_{b_1 b_2 \cdots b_{2s}} \quad .$$

Using Eqs. (II.9) we can see that φ and $\varphi^{(2s-1)}$ satisfies Weyl's equations (II.12):

$$(-i\sigma^0\partial_0 - i\vec{\sigma}\cdot\vec{\nabla}) \otimes 1 \otimes \cdots \otimes 1 \, \varphi(x) = 0 \qquad \text{(II.12a)}$$

$$(-i\sigma^0\partial_0 + i\vec{\sigma}\cdot\vec{\nabla}) \otimes 1 \otimes \cdots \otimes 1 \, \varphi^{(2s-1)}(x) = 0 \qquad \text{(II.12b)}$$

where φ and $\varphi^{(2s-1)}$ are symmetric two component spinors of rank 2s.
Eq. (II.12b) can be derived from (II.12a) by space reflection, so one can consider just one of the Eq. (II.12a) or (II.12b).

The other components φ', ..., $\varphi^{(2s-1)}$ satisfy equations analogous to Weyl's.

III – B.W. THEORY FOR MASSIVE SPIN 1 PARTICLES

This chapter is introduced in order to show how one works with the B.W. theory to describe spin 1 particles. We will see, in an explicit way, that the method is, formally, richer than the usual one in the sense that it allows us to write equations in terms of "observables" (Fields analogous to $\vec{E}$ and $\vec{H}$ in Maxwell's theory) and leads in a straightforward way to the subsidiary condition $\partial^\mu B_\mu = 0$.

Although the method is formally better it is just a little bit trickier, as we will see, to get the physics out of B.W. method.

Free field

Within the B.W. method a spin 1 massive particle of mass m is described, in the non interacting case, by a rank 2 symmetric spinor $\psi_{a_1 a_2}(x)$ obeying a system of two Dirac type equations:

$$\left.\begin{aligned} (i\not\partial \otimes I)\,\psi &= m\psi \\ (I \otimes i\not\partial)\,\psi &= m\psi \quad . \end{aligned}\right. \qquad \text{(III.1)}$$

Equations (III.1) may be derived from the following lagrangian[6)]

$$\mathcal{L}_0 = \bar{\psi} \left\{ \frac{i}{2} [\gamma^\mu \otimes I + I \otimes \gamma^\mu]\, \partial_\mu - m\, I \otimes I \right\} \psi \quad . \qquad \text{(III.2)}$$

If we treat the field $\psi_{a_1 a_2}$ as the independent variable, one gets in particular:

$$\{i\not\partial - m\}_{a_1 a'_s} \psi_{a'_1 a_2}(x) = 0 \quad . \qquad \text{(III.3)}$$

We replace $\psi_{a_1' a_2}$ in (III.3) by its decomposition (II.2) and obtain

$$i\,\gamma^\alpha_{a_1 a_1'}\,\partial_\alpha \left\{B_\mu(x)\,(\gamma^\mu C)_{a_1' a_2} - \frac{1}{2m}\,G_{\mu\nu}(x)\,(\sigma^{\mu\nu}C)_{a_1' a_2}\right\}$$

$$= m\left\{B_\mu(x)\,(\gamma^\mu C)_{a_1 a_2} - \frac{1}{2m}\,G_{\mu\nu}(x)\,(\sigma^{\mu\nu}C)_{a_1 a_2}\right\} \quad . \qquad \text{(III.4)}$$

In order to see that (III.4) leads to the usual equations one has to make some simple operations involving γ matrices. For instance, if one multiplies (III.4) by $(C^{-1})_{a_2 a_1}$ and sums over $a_1 a_2$ one gets:

$$i\,\partial_\alpha B_\mu\,\mathrm{Tr}(\gamma^\alpha \gamma^\mu) = 0$$

from which it follows that:

$$\partial_\mu B^\mu = 0 \quad . \qquad \text{(III.5)}$$

It is interesting to see that the subsidiary condition $\partial_\mu B^\mu = 0$ follows directly from BW equation. If, on the other hand, one multiplies (III.4) by $(C^{-1}\gamma^\beta)_{a_2 a_1}$ one gets:

$$-\frac{i}{2m}\,\partial_\alpha G_{\mu\nu}\,\mathrm{Tr}(\gamma^\alpha \sigma^{\mu\nu} \gamma^\beta) = m\,B_\mu\,\mathrm{Tr}(\gamma^\mu \gamma^\beta)$$

from which one gets

$$-\,\partial^\mu G_{\mu\beta} + m^2 B_\beta = 0 \quad . \qquad \text{(III.6)}$$

Finally if we multiply (III.4) by $(C^{-1}\gamma^\beta\gamma^\lambda)_{a_2 a_1}$ one gets:

$$i\,\partial_\alpha B_\mu\,\mathrm{Tr}(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\lambda) = -\frac{1}{2m}\,G_{\mu\nu}\,\mathrm{Tr}(\sigma^{\mu\nu} \gamma^\beta \gamma^\lambda)$$

from which it follows that

$$\partial_\alpha B_\mu\, g^{\beta\lambda} + \partial^\lambda B^\beta - \partial^\beta B^\lambda = G^{\beta\lambda} \quad . \qquad \text{(III.7)}$$

By using (III.5) in (III.7) one gets

$$G^{\beta\lambda} = \partial^\lambda B^\beta - \partial^\beta B^\lambda \qquad \text{(III.8)}$$

that is, in the decomposition (II.2) of the field ψ, the only acceptable tensor is $\partial^\nu B^\mu - \partial^\mu B^\nu$.

We have then seen that BW equations leads to the following restrictions upon the fields B^μ and $G^{\mu\nu}$:

$$\partial^\mu B_\mu = 0$$

$$G_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu \qquad \text{(III.9)}$$

$$-\,\partial^\mu G_{\mu\nu} + m^2 B_\nu = 0 \quad .$$

In order to see the equivalence between the BW method and the usual approach, in which we assign to a vector field B_μ a spin 1 particle, let us now write $\mathcal{L}_0$ in terms of B_μ.

By using decomposition (II.2) one gets:

$$\mathcal{L}_0 = 4m^2 B^{*\mu} B_\mu - 2\,G^{*\mu\nu} G_{\mu\nu} \qquad \text{(III.10)}$$

or also using (III.8)

$$\mathcal{L}_0 = 4m^2 B^{*\mu} B_\mu + 4\,\partial^\mu B^{*\nu}(\partial_\nu B_\mu - \partial_\mu B_\nu) \quad . \qquad \text{(III.11)}$$

If we consider B_μ as the independent field then one gets from (III.11) the usual Euler–Lagrange for the B_μ field that is:

$$m^2 B_\nu - \partial^\mu G_{\mu\nu} = 0 \tag{III.12}$$

where $$G_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu .$$

Eq. (III.12) is the same that eq. (III.6).

Interacting fields

Let us consider the interaction of massive spin 1 particles with massive spin $\frac{1}{2}$ particles, described, as usual, by rank 1 spinor field η.

If we restrict ourselves to lagrangians that are linear in the ψ fields (that leads, ultimately, to renormalizable models) then, the forms that are compatible with Lorentz invariance assume the form (III.13):

$$\mathcal{L}^I_{int} = g_1\, \bar\psi_{a_1 a_2}\, \eta^c_{a_1}\, \eta^c_{a_2} + g_2\, \bar\psi_{a_1 a_2}\, \eta^c_{a_1}\, \eta^c_{a_2} + \text{h.c.} = g_1\, \eta^c\, \bar\psi\, \eta + g_2\, \eta\, \bar\psi\, \eta + \text{h.c.} \tag{III.13}$$

where g_1 and g_2 are constants with dimension $[L]^{1/2} = \dfrac{1}{[M]^{1/2}}$.

In the following we shall take $g_2 = 0$. At least in the zero mass limit nature prefers this type of coupling.

We shall study the following total lagrangean for a spin 1 massive field interacting with a spin $\frac{1}{2}$ massive field:

$$\mathcal{L} = \bar\psi \left\{ \frac{i}{2}(\gamma^\mu \otimes I + I \otimes \gamma^\mu)\partial_\mu - m\, I \otimes I \right\} \psi + g_1\, \eta^c\, \bar\psi\, \eta + \bar\eta\, (i\partial\!\!\!/ - m_f)\, \eta . \tag{III.14}$$

Replacing ψ by its decomposition (II.2) we obtain for $\mathcal{L}$

$$\mathcal{L} = 4m^2 B^{*\mu} B_\mu - 2\, G^{*\mu\nu} G_{\mu\nu} - g_1 \sqrt{m} \left\{ B^{*\mu}\, \bar\eta\, \gamma^\mu\, \eta - \frac{1}{2m} G^*_{\mu\nu}\, \bar\eta\, \sigma^{\mu\nu} \eta \right\} + \bar\eta\, (i\partial\!\!\!/ - m_f)\, \eta . \tag{III.15}$$

Using $G^{\mu\nu} = \partial^\nu B^\mu - \partial^\mu B^\nu$ we can write $\mathcal{L}$ in the following form

$$\mathcal{L}_0 = 4m^2 B^{*\mu} B_\mu + 4\, \partial^\mu B^{*\nu}(\partial_\nu B_\mu - \partial_\mu B_\nu) -$$
$$- g_1 \sqrt{m} \left\{ B^*_\nu\, \bar\eta\, \gamma^\nu\, \eta + \frac{1}{m}\, \partial_\mu B^*_\nu (\bar\eta\, \sigma^{\mu\nu} \eta) \right\} + \bar\eta\, (i\partial\!\!\!/ - m_f)\, \eta .$$

Writing now Lagranges equations with respect to B^ν and η

$$\frac{\partial \mathcal{L}}{\partial B^{*\nu}} - \partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu B^{*\nu})} \right] = 0 \tag{III.16}$$

$$\frac{\partial \mathcal{L}}{\partial \eta_a} - \partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_a)} \right] = 0 . \tag{III.17}$$

we obtain from (III.16)

$$4m^2 B_\nu - g_1 \sqrt{m}\, \bar\eta\, \gamma_\nu\, \eta - \partial^\mu \left\{ 4\, (\partial_\nu B_\mu - \partial_\mu B_\nu) - \frac{g_1}{\sqrt{m}}\, \bar\eta\, \sigma_{\mu\nu}\, \eta \right\} = 0$$

or also remembering that $G_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu$

$$- \partial^\mu G_{\mu\nu} + m^2 B_\nu = \frac{g_1 \sqrt{m}}{4}\, \bar\eta\, \gamma_\nu\, \eta - \frac{g_1}{4\sqrt{m}}\, \partial^\mu (\bar\eta\, \sigma_{\mu\nu}\, \eta) . \tag{III.18}$$

On the other hand, we obtain from (III.17)

$$\left\{ i\partial\!\!\!/ - m_f \right\}_{a_1a_2}\eta_{a_2} = 2g_1\sqrt{m}\left\{ \gamma_\mu B^\mu - \frac{1}{2m}G_{\mu\nu}\sigma^{\mu\nu} \right\}_{a_1a_2}\eta_{a_2} . \qquad \text{(III.19)}$$

Treating now $\bar{\psi}_{a_1a_2}$, $\psi_{a_1a_2}$, η as independent fields we obtain using (III.9) and Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}_{a_1a_2}} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_{a_1a_2})}\right] = 0$$

$$\frac{\partial\mathcal{L}}{\partial\eta_a} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\eta_a)}\right] = 0$$

the following equations

$$\frac{i}{2}\left\{\gamma^\mu_{a_1a_1'}\delta_{a_2a_2'} + \delta_{a_1a_1'}\gamma^\mu_{a_2a_2'}\right\}\partial_\mu\psi_{a_1'a_2'} - m\,\psi_{a_1a_2} + g_1\,\eta^c_{a_1}\eta_{a_2} = 0 \qquad \text{(III.20)}$$

$$2g_1\,\eta^c_{a_1}\bar{\psi}_{a_1a_2} - m_f\,\bar{\eta}_{a_2} - i\partial_\mu(\bar{\eta})_{a_1}\gamma^\mu_{a_1a_2} = 0 . \qquad \text{(III.21)}$$

Replacing $\psi_{a_1a_2}$ by its decomposition (II.2) in (III.20) we obtain

$$0 = \frac{i}{2}\partial_\alpha\left\{ B_\mu(\gamma^\alpha\gamma^\mu C)_{a_1a_2} - \frac{1}{2m}G_{\mu\nu}(\gamma^\alpha\sigma^{\mu\nu}C)_{a_1a_2}\right\}$$

$$-\frac{i}{2}\partial_\alpha\left\{ B_\mu(\gamma^\mu\gamma^\alpha C)_{a_1a_2} - \frac{1}{2m}G_{\mu\nu}(\sigma^{\mu\nu}\gamma^\alpha C)_{a_1a_2}\right\}$$

$$-m\left\{ B_\mu(\gamma^\mu C)_{a_1a_2} - \frac{1}{2m}G_{\mu\nu}(\sigma^{\mu\nu}C)_{a_1a_2}\right\} + \frac{g_1}{\sqrt{m}}\eta^c_{a_1}\eta_{a_2}$$

or using (III.9)

$$0 = -\frac{1}{2}G_{\alpha\mu}(\sigma^{\alpha\mu}C)_{a_1a_2} - \frac{1}{2m}\partial_\alpha G_{\mu\nu}\left[(\gamma^\mu g^{\alpha\nu} - \gamma^\nu g^{\alpha\mu})C\right]_{a_1a_2}$$

$$- m\,B_\mu(\gamma^\mu C)_{a_1a_2} + \frac{1}{2m}G_{\mu\nu}(\sigma^{\mu\nu}C)_{a_1a_2} + \frac{g_1}{\sqrt{m}}\eta^c_{a_1}\eta_{a_2} . \qquad \text{(III.22)}$$

Now multipling (III.22) by $(C^{-1}\gamma^\rho)_{a_2a_1}$ we obtain

$$\partial_\mu G^{\rho\mu} = -m^2B^\rho + g_1\frac{\sqrt{m}}{4}\bar{\eta}\gamma^\rho\eta . \qquad \text{(III.23)}$$

Model (III.23) leads to equations analogous to Maxwell's. In fact, in analogy with massless case, we name

$$\begin{aligned} G^{ok} &= E^k \\ G^{jk} &= \epsilon^{ljk}H^l \end{aligned} \qquad \text{(III.24)}$$

we obtain from (III.23) the following set of equations:

$$\begin{aligned} &\vec{\nabla}\cdot\vec{E} + m^2B^0 = g_1\frac{\sqrt{m}}{4}\bar{\eta}\gamma^0\eta \\ &\partial^0\vec{E} - \vec{\nabla}\wedge\vec{H} - m^2\vec{B} = -g_1\frac{\sqrt{m}}{4}\bar{\eta}\gamma^0\eta \\ &\vec{\nabla}\cdot\vec{H} = 0 \\ &\partial^0\vec{H} + \vec{\nabla}\wedge\vec{E} = 0 . \end{aligned} \qquad \text{(III.25)}$$

Eqs. (III.25) are analogous to Maxwell's. That is we get a set of coupled equations of first order in term of the observables $\vec{E}$ and $\vec{H}$.

Finally eq. (III.21) gives rise, after taking the complex conjugated and replace ψ by its decomposition (II.2), to (III.19).

Hamiltonian

We take the form (III.14) of $\mathcal{L}$ as a starting point. The two fields η and ψ are independent. We construct conjugate momenta from $\mathcal{L}$ by the standard prescription; so we obtain

$$\pi_{a_1 a_2} = \frac{\partial \mathcal{L}}{\partial(\partial^0 \psi)_{a_1 a_2}} = \bar{\psi}_{a_1' a_2'} \frac{i}{2} \left\{ \gamma^0_{a_1' a_1} \delta_{a_2' a_2} + \delta_{a_1' a_1} \gamma^0_{a_2' a_2} \right\}$$

$$\pi_{a_1} = \frac{\partial \mathcal{L}}{\partial(\partial^0 \eta)_{a_1}} = \bar{\eta}_{a_2}\, i\, \gamma^0_{a_2 a_1} \quad . \tag{III.26}$$

The hamiltonian is defined by

$$\mathrm{H} = \pi_{a_1 a_2} \partial^0 \psi_{a_1 a_2} + \pi_{a_1} \partial^0 \eta_{a_1} - \mathcal{L} \quad .$$

Replacing $\psi_{a_1 a_2}(\bar{\psi}_{a_1 a_2})$ by its decomposition (II.2) one gets

$$\mathrm{H} = 4\left\{G^{*0\mu}\, \partial_0\, B_\mu - B^*_\mu\, \partial_0\, G^{0\mu}\right\} + \bar{\eta}\,(i\gamma^0\partial_0)\eta - 4\left\{m^2\, B^{*\mu}\, B_\mu - \frac{1}{2}\, G^{*\mu\nu}\, G_{\mu\nu}\right\} +$$

$$+ g_1\sqrt{m}\left\{B^*_\mu\, \bar{\eta}\, \gamma^\mu\, \eta - \frac{1}{2m}\, G^*_{\mu\nu}\, \bar{\eta}\, \sigma^{\mu\nu}\, \eta\right\} - \bar{\eta}\left\{i\gamma^k\partial_k - m_f\right\}\eta \tag{III.27}$$

$\mu, \nu = 0, 1, 2, 3$

$k = 1, 2, 3$.

Or, integrating by parts:

$$\mathrm{H} = 4\left\{G^{*0\mu}\, \partial_0\, B_\mu + \partial_0\, B^*_\mu\, G^{0\mu}\right\} - 4\left\{m^2\, B^{*\mu}\, B_\mu - \frac{1}{2}\, G^{*\mu\nu} G_{\mu\nu}\right\} +$$

$$+ g_1\sqrt{m}\left\{B^*_\mu\, \bar{\eta}\, \gamma^\mu\, \eta - \frac{1}{2m}\, G^*_{\mu\nu}\, \bar{\eta}\, \sigma^{\mu\nu}\, \eta\right\} - \bar{\eta}\left[i\gamma^k\partial_k - m_f\right]\eta \tag{III.29}$$

where $k, j = 1, 2, 3$

$\mu, \nu = 0, 1, 2, 3$.

Now, following Bjorken we adopt the notation

$$E^j = E^j_l + E^j_t = -\partial_j\, B^0 - \partial_0\, B^j$$

and we assume that the fields $\vec{E}$ and $\vec{H}$ are real, so we obtain

$$\mathrm{H} = 4\left[\vec{H}^2 + \vec{E}_t^2\right] - 4\, \vec{E}_l^2 - 4\, m^2\, B^\mu\, B_\mu +$$

$$+ g_1\, \sqrt{m}\left\{B^0\, \bar{\eta}\, \gamma^0\, \eta + B_j\, \bar{\eta}\, \gamma^j\, \eta - \frac{1}{2m}\, G_{\mu\nu}\, \bar{\eta}\, \sigma^{\mu\nu}\, \eta\right\} - \bar{\eta}\left[i\gamma^k\partial_k - m_f\right]\eta$$

$k, j = 1, 2, 3$

$\mu, \nu = 0, 1, 2, 3$.

IV – SPIN 1 MASSLESS PARTICLES

The extension of B.W. theory to massless particles is not straightforward. For example the zero mass limit of the decomposition (II.2) is meaningless. Furthermore, for massless spin 1 particles, the appropriate Lagrangian is not the zero mass limit of (III.2). The appropriate lagrangian in this case is

$$\mathcal{L}_0 = \bar{\psi}\left\{\frac{i}{2}\left[\gamma^\mu \otimes I + I \otimes \gamma^\mu\right]\partial_\mu\right\}\psi \quad . \tag{IV.1}$$

As pointed out in chapter II , for massless spin 1 particles one can work with a rank 2 spinor field ψ but the only relevant combinations are the chiral components

$$\begin{aligned}
\tilde{\psi}_{RR} &= \tfrac{1}{2}(I+\gamma^5)\otimes\tfrac{1}{2}(I+\gamma^5)\,\tilde{\psi} \\
\tilde{\psi}_{RL} &= \tfrac{1}{2}(I+\gamma^5)\otimes\tfrac{1}{2}(I-\gamma^5)\,\tilde{\psi} \\
\tilde{\psi}_{LR} &= \tfrac{1}{2}(I-\gamma^5)\otimes\tfrac{1}{2}(I+\gamma^5)\,\tilde{\psi} \\
\tilde{\psi}_{LL} &= \tfrac{1}{2}(I-\gamma^5)\otimes\tfrac{1}{2}(I-\gamma^5)\,\tilde{\psi} \quad .
\end{aligned} \tag{IV.2}$$

The equations satisfied by these components are

$$\begin{aligned}
(i\partial\!\!\!/\otimes I)\tilde{\psi}_{RR} &= 0 \\
(i\partial\!\!\!/\otimes I)\tilde{\psi}_{RL} &= 0 \\
(i\partial\!\!\!/\otimes I)\tilde{\psi}_{LR} &= 0 \\
(i\partial\!\!\!/\otimes I)\tilde{\psi}_{LL} &= 0 \quad .
\end{aligned} \tag{IV.3}$$

It is simple to check that the lagrangian that gives rise to these equations is:

$$\tilde{\mathcal{L}}_0 = \bar{\tilde{\psi}}_{RL}(i\partial\!\!\!/\otimes I)\tilde{\psi}_{RR} + \bar{\tilde{\psi}}_{LR}(i\partial\!\!\!/\otimes I)\tilde{\psi}_{LL} + \bar{\tilde{\psi}}_{RR}(i\partial\!\!\!/\otimes I)\tilde{\psi}_{RL} + \bar{\tilde{\psi}}_{LL}(i\partial\!\!\!/\otimes I)\tilde{\psi}_{LR} \quad . \tag{IV.4}$$

We will verify that $\tilde{\mathcal{L}}_0$ given in (IV.4) is Lorentz invariant.

By a Poincaré transformation of the coordinates, the fields transforms as below:

$$\begin{aligned}
\tilde{\psi}_{RR}(x) &\longrightarrow \tilde{\psi}'_{RR}(x') = D\otimes D\,\tfrac{1}{2}(I+\det\ell\,\gamma^5)\otimes\tfrac{1}{2}(I+\det\ell\,\gamma^5)\tilde{\psi}(x) \\
\tilde{\psi}_{RL}(x) &\longrightarrow \tilde{\psi}'_{RL}(x') = D\otimes D\,\tfrac{1}{2}(I+\det\ell\,\gamma^5)\otimes\tfrac{1}{2}(I-\det\ell\,\gamma^5)\tilde{\psi}(x) \\
\tilde{\psi}_{LR}(x) &\longrightarrow \tilde{\psi}'_{LR}(x') = D\otimes D\,\tfrac{1}{2}(I-\det\ell\,\gamma^5)\otimes\tfrac{1}{2}(I+\det\ell\,\gamma^5)\tilde{\psi}(x) \\
\tilde{\psi}_{LL}(x) &\longrightarrow \tilde{\psi}'_{LL}(x') = D\otimes D\,\tfrac{1}{2}(I-\det\ell\,\gamma^5)\otimes\tfrac{1}{2}(I-\det\ell\,\gamma^5)\tilde{\psi}(x) \quad .
\end{aligned} \tag{IV.5}$$

So we see that by Poincare coordinate transformation

$$\begin{aligned}
\bar{\tilde{\psi}}_{RL}\, i\partial\!\!\!/\otimes I\,\tilde{\psi}_{RR} &\longrightarrow \bar{\tilde{\psi}}_{RL}\, i\partial\!\!\!/\otimes I\,\tilde{\psi}_{RR} && \text{if } \det\ell = +1 \\
&\longrightarrow \bar{\tilde{\psi}}_{LR}\, i\partial\!\!\!/\otimes I\,\tilde{\psi}_{LL} && \text{if } \det\ell = -1 \\
\bar{\tilde{\psi}}_{RR}\, i\partial\!\!\!/\otimes I\,\tilde{\psi}_{RL} &\longrightarrow \bar{\tilde{\psi}}_{RR}\, i\partial\!\!\!/\otimes I\,\tilde{\psi}_{RL} && \text{if } \det\ell = 1 \\
&\longrightarrow \bar{\tilde{\psi}}_{LL}\, i\partial\!\!\!/\otimes I\,\tilde{\psi}_{LR} && \text{if } \det\ell = -1
\end{aligned} \tag{IV.6}$$

so, $\mathcal{L}_0$ is globaly Lorentz invariant. On the other hand, it is easy to see, remembering that

$$\psi = \psi_{RR} + \psi_{RL} + \psi_{LR} + \psi_{LL} \tag{IV.7}$$

that the lagrangian (IV.4) is equivalent to

$$\ell_0 = \bar{\tilde{\psi}}\,(\mathrm{i}\,\partial\!\!\!/ \otimes \mathrm{I})\,\tilde{\psi}$$

In fact if one substitutes ψ by (IV.7) then the non zero terms of the above lagrangian are those of (IV.4). It is interesting to note the difference between (IV.1) and (IV.4). Although the difference at first sight seems to be just a matter of symmetry, it is a little bit deeper. The difference is, in fact, chiral invariance. In fact lagrangian (IV.4) is invariant under the chiral transformation

$$\tilde{\psi} \longrightarrow \tilde{\psi}' = e^{-\mathrm{i}\alpha\gamma^5} \otimes \mathrm{I}\,\tilde{\psi} \tag{IV.8}$$

whereas in the case of (IV.1) there is no simple extension of the usual chiral transformation to spin 1 particles.

Let us consider now the set of Weyl's equations in the case of spin 1 particles. In this case one writes:

$$\tilde{\psi} \sim \begin{bmatrix} \xi^{\mathrm{I}} \\ \dot{\chi}^{\mathrm{I}} \end{bmatrix} \otimes \begin{bmatrix} \xi^{\mathrm{II}} \\ \dot{\chi}^{\mathrm{II}} \end{bmatrix}$$

so

$$\tilde{\psi}_{\mathrm{RR}} \sim \begin{bmatrix} \xi^{\mathrm{I}} \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \xi^{\mathrm{II}} \\ 0 \end{bmatrix}$$

$$\tilde{\psi}_{\mathrm{RL}} \sim \begin{bmatrix} \xi^{\mathrm{I}} \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \dot{\chi}^{\mathrm{II}} \end{bmatrix}$$

$$\tilde{\psi}_{\mathrm{LR}} \sim \begin{bmatrix} 0 \\ \dot{\chi}^{\mathrm{I}} \end{bmatrix} \otimes \begin{bmatrix} \xi^{\mathrm{II}} \\ 0 \end{bmatrix}$$

$$\tilde{\psi}_{\mathrm{LL}} \sim \begin{bmatrix} 0 \\ \dot{\chi}^{\mathrm{I}} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \dot{\chi}^{\mathrm{II}} \end{bmatrix} .$$

Weyl's equations can be derived in this case, by taking only the $\tilde{\psi}_{\mathrm{RR}}$, $\tilde{\psi}_{\mathrm{LL}}$ components and are

$$\begin{aligned} (-\mathrm{i}\sigma^0\partial_0 - \mathrm{i}\vec{\sigma}\cdot\vec{\nabla})_{\mathrm{aa'}}\, \xi^{\mathrm{I}}_{\mathrm{a'}}\, \xi^{\mathrm{II}}_{\mathrm{b}} &= 0 \\ (-\mathrm{i}\sigma^0\partial_0 + \mathrm{i}\vec{\sigma}\cdot\vec{\nabla})_{\mathrm{aa'}}\, \dot{\chi}^{\mathrm{I}}_{\mathrm{a'}}\, \dot{\chi}^{\mathrm{II}}_{\mathrm{b}} &= 0 \end{aligned} \tag{IV.9}$$

where $\xi^{\mathrm{I}}\,\xi^{\mathrm{II}}$ and $\dot{\chi}^{\mathrm{I}}\,\dot{\chi}^{\mathrm{II}}$ are supposed to be symmetrised in spin indices.

The generalized helicity operator for spin 1 particles is

$$\tfrac{1}{2}(\vec{\Sigma}\cdot\hat{n} \otimes \mathrm{I} + \mathrm{I} \otimes \vec{\Sigma}\cdot\hat{n}) \equiv \mathrm{W} \quad ; \quad \hat{n} = \frac{\vec{p}}{|\vec{p}|} .$$

It is straightforward to show that ψ_{RR}, ψ_{RL}, ψ_{LR}, ψ_{LL} are eigenstates of W with eigenvalues $+1, 0, -1$ that is

$$\tfrac{1}{2}(\vec{\Sigma}\cdot\hat{n} \otimes \mathrm{I} + \mathrm{I} \otimes \vec{\Sigma}\cdot\hat{n})\,\tilde{\psi}_{\mathrm{RR}} = 1$$

$$\tfrac{1}{2}(\vec{\Sigma}\cdot\hat{n} \otimes \mathrm{I} + \mathrm{I} \otimes \vec{\Sigma}\cdot\hat{n})\,\tilde{\psi}_{\mathrm{RL}} = 0$$

$$\tfrac{1}{2}(\vec{\Sigma}\cdot\hat{n} \otimes \mathrm{I} + \mathrm{I} \otimes \vec{\Sigma}\cdot\hat{n})\,\tilde{\psi}_{\mathrm{LR}} = 0$$

$$\tfrac{1}{2}(\vec{\Sigma}\cdot\hat{n} \otimes \mathrm{I} + \mathrm{I} \otimes \vec{\Sigma}\cdot\hat{n})\,\tilde{\psi}_{\mathrm{LL}} = -1 .$$

In order to ilustrate the difficulty with the zero mass limit let us write the expressions for the chiral components in the massive case. By using the decomposition

(II.2) one can write:

$$\psi_{R_{a_1}R_{a_2}} = \frac{1}{2}(I+\gamma^5)_{a_1a_1'}\frac{1}{2}(I+\gamma^5)_{a_2a_2'}\psi_{a_1'a_2'} = \frac{1}{\sqrt{m}}\left[\frac{1}{2}(I+\gamma^5)\sigma^{\mu\nu}c\right]_{a_1a_2}\partial_\mu B_\nu$$

$$\psi_{R_{a_1}L_{a_2}} = \frac{1}{2}(I+\gamma^5)_{a_1a_1'}\frac{1}{2}(I-\gamma^5)_{a_2a_2'}\psi_{a_1'a_2'} = \sqrt{m}\left[\frac{1}{2}(I+\gamma^5)\gamma^\mu c\right]_{a_1a_2}B_\mu$$

$$\psi_{L_{a_1}R_{a_2}} = \frac{1}{2}(I-\gamma^5)_{a_1a_1'}\frac{1}{2}(I+\gamma^5)_{a_2a_2'}\psi_{a_1'a_2'} = \sqrt{m}\left[\frac{1}{2}(I-\gamma^5)\gamma^\mu c\right]_{a_1a_2}B_\mu \tag{IV.10}$$

$$\psi_{L_{a_1}L_{a_2}} = \frac{1}{2}(I-\gamma^5)_{a_1a_1'}\frac{1}{2}(I-\gamma^5)_{a_2a_2'}\psi_{a_1'a_2'} = \frac{1}{\sqrt{m}}\left[\frac{1}{2}(I-\gamma^5)\sigma^{\mu\nu}c\right]_{a_1a_2}\partial_\mu B_\nu \ .$$

From (IV.10) one can see that the zero mass limit leads to chiral components that are zero or ill defined. However, from the requirement that ψ is a symmetric field in its spin indices one can write a decomposition analogous to (II.2) for $\tilde{\psi}$ that is:

$$\tilde{\psi}_{a_1a_2}(x) = c_1 A_\mu(x)(\gamma^\mu C)_{a_1a_2} - c_2 F_{\mu\nu}(x)(\sigma^{\mu\nu}C)_{a_1a_2} \ . \tag{IV.11}$$

Comparing decompositions (IV.11) and (II.2) we see that constants c_1 and c_2 must be respectively of dim $[M]^{1/2}$ and $\frac{1}{[M]^{1/2}}$ and, in case of QED will be determined later. Using Eq. (II.7), we see that we must have

$$i\,\partial_{a_1a_1'}\psi_{a_1'a_2}(x) = 0$$

or using decomposition (IV.11)

$$i(\gamma^\alpha\partial_\alpha)_{a_1a_1'}\left\{c_1 A_\mu(x)(\gamma^\mu C)_{a_1a_2} - c_2 F_{\mu\nu}(x)(\sigma^{\mu\nu}C)_{a_1a_2}\right\} = 0 \ . \tag{IV.12}$$

Multiplying (IV.12) by $(C^{-1})_{a_2a_1}$ we obtain

$$\partial^\mu A_\mu = 0 \tag{IV.13}$$

that is, the field A_μ satisfies the Lorentz condition. Furthermore multipling (IV.12) by $(C^{-1}\gamma^\beta)_{a_2a_1}$ we obtain

$$\partial^\mu F_{\mu\nu} = 0 \ . \tag{IV.14}$$

Writing $F_{\mu\nu}$ as

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\nu A_\mu$$

we will then, get

$$\partial_\lambda\partial^\lambda A^\beta - \partial_\lambda\partial^\beta A^\lambda = 0 \Longrightarrow \Box A^\beta = 0 \ .$$

For a massless field we must have following conditions:

$$\begin{aligned} \partial^\mu A_\mu &= 0 \\ \partial^\mu F_{\mu\nu} &= 0 \\ \Box A_\mu &= 0 \end{aligned} \tag{IV.15}$$

and we note that in this case A_μ and $F_{\mu\nu}$ are, in principle independent fields.

The most general form for $F_{\mu\nu}$ is

$$F_{\mu\nu} = f_{\mu\nu} \pm i\tilde{f}_{\mu\nu}$$

where $\tilde{f}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\rho} f^{\alpha\beta}$.

By using the decomposition (IV.11) we get the following expressions for $\psi_{RR}, \psi_{RL}, \psi_{LR}, \psi_{LL}$

$$\begin{aligned}
\tilde{\psi}_{R_{a_1}R_{a_2}} &= \frac{1}{2}(I+\gamma^5)_{a_1a_1'}\frac{1}{2}(I+\gamma^5)_{a_2a_2'}\tilde{\psi}_{a_1'a_2'} = -c_2\left[\frac{1}{2}(I+\gamma^5)\sigma^{\mu\nu}C\right]_{a_1a_2}F_{\mu\nu} \\
\tilde{\psi}_{R_{a_1}L_{a_2}} &= \frac{1}{2}(I+\gamma^5)_{a_1a_1'}\frac{1}{2}(I-\gamma^5)_{a_2a_2'}\tilde{\psi}_{a_1'a_2'} = c_1\left[\frac{1}{2}(I+\gamma^5)\gamma^{\mu}C\right]_{a_1a_2}A_{\mu} \\
\tilde{\psi}_{L_{a_1}R_{a_2}} &= \frac{1}{2}(I-\gamma^5)_{a_1a_1'}\frac{1}{2}(I+\gamma^5)_{a_2a_2'}\tilde{\psi}_{a_1'a_2'} = c_1\left[\frac{1}{2}(I-\gamma^5)\gamma^{\mu}C\right]_{a_1a_2}A_{\mu} \\
\tilde{\psi}_{L_{a_1}L_{a_2}} &= \frac{1}{2}(I-\gamma^5)_{a_1a_1'}\frac{1}{2}(I-\gamma^5)_{a_2a_2'}\tilde{\psi}_{a_1'a_2'} = -c_2\left[\frac{1}{2}(I-\gamma^5)\sigma^{\mu\nu}C\right]_{a_1a_2}F_{\mu\nu} .
\end{aligned} \tag{IV.16}$$

As a final result we will show that the right–right and the left–left components, or, equivalently Weyls components give rise to Maxwell's equations.

For this purpose we will use notation (II.11) which gives the expression of $\tilde{\psi}_{RR}\ (\tilde{\psi}_{LL})$ in terms of two components spinors.

One has $\quad \tilde{\psi}_{RR} = \begin{bmatrix} \varphi \\ 0 \\ 0 \\ 0 \end{bmatrix} \left[\tilde{\psi}_{LL} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varphi^{(3)} \end{bmatrix} \right].$

The symmetry of φ allows us to write the following decomposition

$$\varphi(x) = \vec{f}(x)\cdot\vec{\sigma}\, C_1 \tag{IV.17}$$

from we get the following equations for the vector $\vec{f}(x)$

$$\begin{cases} \vec{\nabla}\cdot\vec{f} = 0 \\ i\partial^0\vec{f} = \vec{\nabla}\wedge\vec{f} \ . \end{cases} \tag{IV.18}$$

Now if we define $\quad \mathcal{F}^{\mu\nu} = F^{\mu\nu} - i\tilde{F}^{\mu\nu}$

where
$$\begin{cases} F^{\mu\nu} = \partial^{\nu}A^{\mu} - \partial^{\mu}A^{\nu} \\ \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \ . \end{cases}$$

A^{μ} being the vector potential and $\tilde{F}^{\mu\nu}$ obeing $\partial^{\mu}\tilde{F}_{\mu\nu} = \partial^{\nu}\tilde{F}_{\mu\nu} = 0$ and if

$$f_{\mu}(x) = \frac{1}{2}\epsilon_{0\mu\alpha\beta}\mathcal{F}^{\alpha\beta}(x) = \frac{1}{2}\epsilon_{\mu\alpha\beta}\mathcal{F}^{\alpha\beta}(x) \tag{IV.19}$$

$$\mu, \alpha, \beta = 1, 2, 3$$

equations (IV.18) for $f_{\mu}(x)$ imply Maxwell's equations that is, equations (IV.18) imply:

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{H} = 0 \\ \vec{\nabla}\wedge\vec{E} + \partial_0\vec{H} = 0 \\ \vec{\nabla}\wedge\vec{H} - \partial_0\vec{E} = 0 \end{cases} \tag{IV.20}$$

where
$$\begin{cases} \vec{E} = -\partial_0\vec{A} - \vec{\nabla} A = F_0{}^{\mu} \\ \vec{H} = \vec{\nabla}\wedge\vec{A} = \tilde{F}_0{}^{\mu} \ . \end{cases}$$

V – INTERACTING FIELDS – QED

Let us consider now the interaction of massless spin 1 fields. We will be mainly concerned with the interaction of these particles with ordinary matter. That is we will be concerned with the most general interaction lagrangian describing massless spin 1 particles interacting with spin $\frac{1}{2}$ massive (m_f) particles.

By imposing that the lagrangian is linear in the ψ fields, then the most general form that is bilinear in the matter field η that we can construct with the four independent fields ψ_{RR}, ψ_{RL}, ψ_{LR} and ψ_{LL} is:

$$\mathcal{L}_{int} = A\bar{\tilde{\psi}}_{R_{a_1}R_{a_2}}\eta_{a_1}\eta_{a_2} + B\bar{\tilde{\psi}}_{L_{a_1}L_{a_2}}\eta_{a_1}\eta_{a_2} + D\bar{\tilde{\psi}}_{R_{a_1}L_{a_2}}\eta_{a_1}\eta_{a_2} + E\bar{\tilde{\psi}}_{L_{a_1}R_{a_2}}\eta_{a_1}\eta_{a_2} +$$

$$+ F\bar{\tilde{\psi}}_{R_{a_1}R_{a_2}}\eta^c_{a_1}\eta_{a_2} + J\bar{\tilde{\psi}}_{L_{a_1}L_{a_2}}\eta^c_{a_1}\eta_{a_2} + K\bar{\tilde{\psi}}_{R_{a_1}L_{a_2}}\eta^c_{a_1}\eta_{a_2} + L\bar{\tilde{\psi}}_{L_{a_1}R_{a_2}}\eta^c_{a_1}\eta_{a_2} + \text{h.c.} \qquad \text{(V.1)}$$

where A, B, D, E, F, J, K, L are arbitrary constants.

The reason for so many terms is that we shall assume, to start with, that ordinary matter couples with different coupling to the chiral components of the field ψ. As a matter of fact we shall see that, in this case, only the zero helicity components couple with ordinary matter.

The number of coupling constants can be reduced to four by imposing Lorentz invariance. In fact using the transformation rules (IV.6) of the fields ψ_{RR}, ψ_{RL}, ψ_{LR} and ψ_{LL} one can see that, under Lorentz transformations, one has:

$$\begin{aligned} \bar{\tilde{\psi}}_{RR}\,\eta\eta\,(\tilde{\psi}_{RR}\,\eta^c\eta) &\longrightarrow \bar{\tilde{\psi}}_{RR}\,\eta\eta\,(\bar{\tilde{\psi}}\,\eta^c\eta) \qquad \text{if} \quad \det \ell = 1 \\ &\longrightarrow \bar{\tilde{\psi}}_{LL}\,\eta\eta\,(\bar{\tilde{\psi}}\,\eta^c\eta) \qquad \text{if} \quad \det \ell = -1 \end{aligned} \qquad \text{(V.2)}$$

$$\begin{aligned} \bar{\tilde{\psi}}_{RL}\,\eta\eta\,(\bar{\tilde{\psi}}_{RL}\,\eta^c\eta) &\longrightarrow \bar{\tilde{\psi}}_{RL}\,\eta\eta\,(\bar{\tilde{\psi}}_{RL}\,\eta^c\eta) \qquad \text{if} \quad \det \ell = 1 \\ &\longrightarrow \bar{\tilde{\psi}}_{LR}\,\eta\eta\,(\bar{\tilde{\psi}}_{LR}\,\eta^c\eta) \qquad \text{if} \quad \det \ell = -1 \ . \end{aligned}$$

So that in order to insure Lorentz invariance one has to require that

$$\begin{aligned} A &= B \\ D &= E \\ F &= J \\ K &= L \end{aligned}$$

The most general interaction lagrangian is then:

$$\mathcal{L}_{int} = A(\bar{\tilde{\psi}}_{RR}\,\eta\eta + \bar{\tilde{\psi}}_{LL}\,\eta\eta) + D(\bar{\tilde{\psi}}_{RL}\,\eta\eta + \bar{\tilde{\psi}}_{LR}\,\eta\eta) +$$

$$+ F(\bar{\tilde{\psi}}_{RR}\,\eta^c\eta + \bar{\tilde{\psi}}_{LL}\,\eta^c\eta) + K(\bar{\tilde{\psi}}_{RL}\,\eta^c\eta + \bar{\tilde{\psi}}_{LR}\,\eta^c\eta) \ . \qquad \text{(V.3)}$$

We now consider the particular lagrangian of interaction obtained for $A = D = K = 0$

$$\mathcal{L}_{int} = F(\bar{\tilde{\psi}}_{RR}\,\eta^c\eta + \bar{\tilde{\psi}}_{LL}\,\eta^c\eta) \ . \qquad \text{(V.4)}$$

The total lagrangian is then the following

$$\mathcal{L} = \bar{\bar{\psi}}_{RL}(i\not{\partial}\otimes I)\,\bar{\bar{\psi}}_{RR} + \bar{\bar{\psi}}_{LR}(i\not{\partial}\otimes I)\,\bar{\bar{\psi}}_{LL} + \bar{\bar{\psi}}_{RR}(i\not{\partial}\otimes I)\,\bar{\bar{\psi}}_{RL} + \bar{\bar{\psi}}_{LL}(i\not{\partial}\otimes I)\,\bar{\bar{\psi}}_{LR} +$$

$$+ F(\bar{\bar{\psi}}_{RR}\,\eta^c\eta + \bar{\bar{\psi}}_{LL}\,\eta^c\eta) + \bar{\eta}\,(i\not{\partial} - m_f)\eta \ . \tag{V.5}$$

Writing the Lagrange equations explicity one has

$$i\not{\partial}\otimes I\,\bar{\bar{\psi}}_{LR} + F\eta^c\eta = 0 \tag{V.6a}$$

$$i\not{\partial}\otimes I\,\bar{\bar{\psi}}_{RL} + F\eta^c\eta = 0 \tag{V.6b}$$

$$i\not{\partial}\otimes I\,\bar{\bar{\psi}}_{RR} = 0 \tag{V.6c}$$

$$i\not{\partial}\otimes I\,\bar{\bar{\psi}}_{LL} = 0 \ . \tag{V.6d}$$

Eqs. (V.6a) and (V.6b) imposes in particular the following restriction

$$2i\,c_1\,\partial^\mu A_\mu = F\,\bar{\eta}\,\eta$$

$$\partial^\mu F_{\mu\nu} = 0 \ .$$

In this case we do not get QED.

QED

Let us consider now the following interaction lagrangian for the spin 1 massless particles and the usual matter

$$\mathcal{L}_{int} = K\left\{ \bar{\bar{\psi}}_{RL}\,\eta^c\,\eta + \bar{\bar{\psi}}_{LR}\,\eta^c\,\eta \right\} \ . \tag{V.7}$$

By adding the free field lagrangians we end up with the following total lagrangian

$$\mathcal{L} = \bar{\bar{\psi}}_{RL}(i\not{\partial}\otimes I)\,\bar{\bar{\psi}}_{RR} + \bar{\bar{\psi}}_{LR}(i\not{\partial}\otimes I)\,\bar{\bar{\psi}}_{LL} + \bar{\bar{\psi}}_{RR}(i\not{\partial}\otimes I)\,\bar{\bar{\psi}}_{RL} + \bar{\bar{\psi}}_{LL}(i\not{\partial}\otimes I)\,\bar{\bar{\psi}}_{LR} +$$

$$+ K\left\{ \bar{\bar{\psi}}_{RL}\,\eta^c\eta + \bar{\bar{\psi}}_{LR}\,\eta^c\eta \right\} + \bar{\eta}\,(i\not{\partial} - m_f)\eta \ . \tag{V.8}$$

One of the most interesting aspects of this approach is that although only some chiral components couple with the usual matter, all chiral components should be considered as dynamical variables. That is one should write five Euler–Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\bar{\bar{\psi}}_{RR}} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\bar{\psi}}_{RR})}\right] = 0$$

$$\frac{\partial\mathcal{L}}{\partial\bar{\bar{\psi}}_{RL}} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\bar{\psi}}_{RL})}\right] = 0$$

$$\frac{\partial\mathcal{L}}{\partial\bar{\bar{\psi}}_{LR}} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\bar{\psi}}_{LR})}\right] = 0$$

$$\frac{\partial\mathcal{L}}{\partial\bar{\bar{\psi}}_{LL}} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\bar{\psi}}_{LL})}\right] = 0$$

$$\frac{\partial\mathcal{L}}{\partial\eta} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\eta)}\right] = 0 \ .$$

Equations (V.9) and $\mathcal{L}$ given by (V.8) leads to:

$$i\partial\!\!\!/\otimes I\,\tilde{\psi}_{RL} = 0 \qquad \text{(V.10a)}$$

$$i\partial\!\!\!/\otimes I\,\tilde{\psi}_{RR} + K\,\eta^c\,\eta = 0 \qquad \text{(V.10b)}$$

$$i\partial\!\!\!/\otimes I\,\tilde{\psi}_{LL} + K\,\eta^c\,\eta = 0 \qquad \text{(V.10c)}$$

$$i\partial\!\!\!/\otimes I\,\tilde{\psi}_{LR} = 0 \qquad \text{(V.10d)}$$

$$2K\,\eta^c\left\{\bar{\tilde{\psi}}_{RL} + \bar{\tilde{\psi}}_{LR}\right\} - m_f\,\bar{\eta} - i\partial_\mu\,(\bar{\eta})\,\gamma^\mu = 0 \ . \qquad \text{(V.10e)}$$

From Eq (V.10e) it follows that

$$2K\,\eta^c_{a_1}\left\{\bar{\tilde{\psi}}_{RL} + \bar{\tilde{\psi}}_{LR}\right\}_{a_1a_2} - m_f\,\bar{\eta}_{a_2} - i\,\partial_\mu\,\bar{\eta}_{a_1}\,\gamma^\mu_{a_1a_2} = 0 \ .$$

Or replacing $\bar{\tilde{\psi}}_{RL}$ and $\bar{\tilde{\psi}}_{LR}$ by expression (IV.16)

$$-2K\,c_1\,\bar{\eta}_{a_1}\,A^*_\mu(\gamma^\mu)_{a_1a_2} - m_f\,\bar{\eta}_{a_2} - i\,\partial_\mu\,\bar{\eta}_{a_1}\,\gamma^\mu_{a_1a_2} = 0 \ . \qquad \text{(V.11)}$$

Taking the hermitian conjugate of (V.11) one gets

$$(i\partial\!\!\!/ - m_f)_{a_1a_2}\,\eta_{a_2} = 2K\,c_1(\gamma^\mu)_{a_1a_2}\,A_\mu\,\eta_{a_2} \ . \qquad \text{(V.12)}$$

Note that by imposing $2K\,c_1 = e$ we get the usual minimal coupling.

Now adding (V.10b) and (V.10c) and replacing $\tilde{\psi}_{RR}$ and $\tilde{\psi}_{LL}$ by there expressions (IV.16) we get

$$-i\,c_2(\gamma^\alpha\sigma^{\mu\nu}C)_{a_1a_2}\,\partial_\alpha\,F_{\mu\nu} + 2K\,\eta^c_{a_1}\eta_{a_2} = 0 \ . \qquad \text{(V.13)}$$

Multipling (V.13) by $(C^{-1}\gamma^\beta)_{a_2a_1}$ we get:

$$\partial_\mu\,F^{\beta\mu} = \frac{K}{4c_2}\,\bar{\eta}\,\gamma^\beta\,\eta \ . \qquad \text{(V.14)}$$

We see that equations (V.14) can be obtained from equations (III.18).

Similarly adding (V.10a) and (V.10d) and replacing $\tilde{\psi}_{LR}$ and $\tilde{\psi}_{RL}$ by expressions (IV.16) we get:

$$i\,c_1(\gamma^\alpha\gamma^\mu C)_{a_1a_2}\,\partial_\alpha\,A_\mu = 0 \ . \qquad \text{(V.15)}$$

So multipling (V.15) by $(C^{-1})_{a_2a_1}$ we have:

$$\partial^\mu A_\mu = 0 \qquad \text{(V.16)}$$

that is, one gets Lorentz condition.

We have seen in IV that the most general form $F_{\mu\nu}$ is

$$F_{\mu\nu} = f_{\mu\nu} + i\tilde{f}_{\mu\nu} \ . \qquad \text{(V.17)}$$

Replacing $F_{\mu\nu}$ by (V.17) in Eq. (V.14) we get an alternative form for Eq. (V.14) that is

$$\begin{cases} \partial_\mu \, f^{\beta\mu} = \frac{K}{4c_2} \, \bar{\eta} \, \gamma^\beta \, \eta \\ \partial_\mu \, \tilde{f}^{\beta\mu} = 0 \end{cases} \qquad \text{(V.18)}$$

These are Maxwell's equations in Lorentz gauge if one writes

$$f_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

and imposes the restriction

$$\frac{K}{4c_2} = e \ .$$

In order to see this equivalence in terms of the observables $\vec{E}$ and $\vec{H}$ one writes as usual

$$f^{ok} = E^k \qquad k, j, l = 1, 2, 3 \qquad f^{jk} = \epsilon^{ljk} H^l \ .$$

Equations (V.18) give rise to the following equations

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= e \, \bar{\eta} \, \gamma^0 \, \eta \\ \partial^0 \vec{E} - \vec{\nabla} \wedge \vec{H} &= -e \, \bar{\eta} \, \vec{\gamma} \, \eta \\ \vec{\nabla} \cdot \vec{H} &= 0 \\ \partial^0 \vec{H} + \vec{\nabla} \wedge \vec{E} &= 0 \end{aligned} \qquad \text{(V.19)}$$

(V.19) are exactly Maxwell's equations.

The total lagrangian as a function of A_μ and $F_{\mu\nu}$

Replacing $\tilde{\psi}_{RR}$, $\tilde{\psi}_{RL}$, $\tilde{\psi}_{LR}$ and $\tilde{\psi}_{LL}$ by the expressions (IV.16) we obtain for the free spin 1 massless lagrangian

$$\mathcal{L} = \bar{\tilde{\psi}}_{RL}(i\partial\!\!\!/ \otimes I) \, \tilde{\psi}_{RR} + \bar{\tilde{\psi}}_{LR}(i\partial\!\!\!/ \otimes I) \, \tilde{\psi}_{LL} + \bar{\tilde{\psi}}_{RR}(i\partial\!\!\!/ \otimes I) \, \tilde{\psi}_{RL} + \bar{\tilde{\psi}}_{LL}(i\partial\!\!\!/ \otimes I) \, \tilde{\psi}_{LR} =$$

$$= 4c_1c_2 \, F^{\mu\nu} F_{\mu\nu} - Kc_1 \, A^*_\mu \, \bar{\eta} \, \gamma^\mu \, \eta + \bar{\eta} \, (i\partial\!\!\!/ - m_f)\eta \ , \qquad \text{(V.21)}$$

Note that the product c_1c_2 is dimensionless and we impose now that

$$c_1 \, c_2 = \frac{1}{4} \ .$$

A_μ and $F_{\mu\nu}$ as independent fields

Replacing $\tilde{\psi}_{RR}$, $\tilde{\psi}_{RL}$, $\tilde{\psi}_{LR}$ and $\tilde{\psi}_{LL}$ by the expressions (IV.16) we obtain for the free spin 1 massless lagrangean (IV.4)

$$\mathcal{L}_0 = 4c_1c_2 \left\{ F^*_{\mu\nu} \left[\partial^\nu A^\mu - \partial^\mu A^\nu \right] - A^*_\mu \left[\partial^\nu F_\nu{}^\mu - \partial^\nu F^\mu{}_\nu \right] \right\}$$

where we have considered A^μ and $F^{\mu\nu}$ as independent fields.

The total lagrangian (V.8) may be written

$$\mathcal{L} = 4c_1c_2 \, F^*_{\mu\nu} \left[\partial^\nu A^\mu - \partial^\mu A^\nu \right] + 4c_1c_2 \, A^*_\mu \left[\partial^\nu F_\nu{}^\mu - \partial^\nu F^\mu{}_\nu \right] -$$

$$- Kc_1 \, A^*_\mu \, \bar{\eta} \, \gamma^\mu \, \eta + \bar{\eta} \, (i\partial\!\!\!/ - m_f)\eta \ . \qquad \text{(V.22)}$$

Writing Lagrange's equations for A^μ, $F^{\mu\nu}$ and η we obtain

$$\frac{\partial \mathcal{L}}{\partial A^{*\nu}} - \partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^{*\nu})} \right] = 0 \Longrightarrow \partial^\nu F^\mu{}_\nu = \frac{K}{8c_2} \bar{\eta} \gamma^\mu \eta \tag{V.23}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\eta}} - \partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \bar{\eta})} \right] = 0 \Longrightarrow (i\partial\!\!\!/ - m_f)\eta = K c_1 A^* \eta \ . \tag{V.24}$$

VI – QUANTIZATION

In this chapter we will consider the quantization of massive fields within the BW theory. We propose also an extension to the zero mass case. We show that there is no basic distinction between this and the usual approach. The only interesting point is that it is possible to quantize the BW fields.

We quantize the BW fields by imposing the following commutation rules for BW's massive fields:

$$\left[\psi_{a_1 a_2 \ldots a_{2s}}(x), \psi_{a_1' a_2' \ldots a_{2s}'}(y) \right]_s =$$

$$= (-i)^{2s-1} \kappa \sum_{\mathcal{P}} i(i\partial\!\!\!/_x + m)_{a_1 a_1'} \cdots i(i\partial\!\!\!/_x + m)_{a_{2s} a_{2s}'} \, \Delta(x-y) \tag{VI.1}$$

where $a_1, \cdots a_1', \cdots = 1, 2, 3, 4$, $\Delta(x-y)$ is the Jordan Pauli function, κ is a constant to be determined, $\mathcal{P}$ denotes all possible permutations among the spinor indices and where we use below convention

$$[\phi, \psi]_s = \phi \psi + (-1)^{2s-1} \phi \psi \tag{VI.2}$$

s being the spin of considered field.

One writes the BW fields as a linear combination of the symmetric matrices of spinor–space and introduce so new fields which are the coefficients of this expansion. Imposing commutation rules (VI.1) and replacing the BW fields by the precedent expansion, we obtain commutation rules for these new fields.

For a massive spin 1 particle we adopt the decomposition of the BW field in spinor space:

$$\psi_{a_1 a_2}(x) = \sqrt{m} \left\{ B_\mu(x) (\gamma^\mu C)_{a_1 a_2} - \frac{1}{2m} G_{\mu\nu}(x) (\sigma^{\mu\nu} C)_{a_1 a_2} \right\} .$$

By writing the plane wave expansion of the new field B_μ as:

$$B_\mu(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_\lambda \frac{1}{2p^0} \left\{ A(p,\lambda)\, \varepsilon^\mu(p,\lambda)\, e^{-ipx} + B^{+\mu}(p,\lambda)\, \varepsilon^{+\mu}(p,\lambda)\, e^{+ipx} \right\} \tag{VI.3}$$

the properties of the polarization ε^μ (deduced from the propertie (III.5) of the B^μ field) are:

$$p^\mu \varepsilon_\mu(p,\lambda) = 0 \quad . \tag{VI.4}$$

Adopting the normalization

$$\varepsilon^{+\mu}(p,\lambda)\, \varepsilon_\mu(p,\lambda') = -\delta_{\lambda\lambda'} \tag{VI.5}$$

we obtain the sum over the polarizations

$$X^{\mu\nu}(p) = \sum_\lambda \varepsilon^\mu(p,\lambda)\, \varepsilon^{+\nu}(p,\lambda) = -\left[g^{\mu\nu} - p^\mu \frac{p^\nu}{m^2} \right] \quad . \tag{VI.6}$$

Finally we can deduce the Feynman propagator for the ψ field

$$i\, D_F^{\mu\nu}(p) = \frac{i}{p^2 - m^2 + i\epsilon} X^{\mu\nu}(p) \quad . \tag{VI.7}$$

Massless BW fields[5)]

As shown in chapter II, for a massless field the fundamental thinks are the two components spinors of rank 2s, $\varphi, \varphi', \cdots \varphi^{(2s-1)}$.

For the massless case we quantize the two component Weyl's spinor φ (II.11) assuming commutation rules (VI.8).

$$\left[\varphi_{b_1 \ldots b_{2s}}(x), \varphi^+_{b'_1 \ldots b'_{2s}}(y) \right]_s =$$

$$= (-i)^{2s-1} \kappa \sum_{\mathcal{P}} i(i\,\sigma^0 \partial_0 - i\,\vec{\sigma}\cdot\vec{\nabla})_{b_1 b'_1} \cdots i(i\,\sigma^0 \partial_0 - i\,\vec{\sigma}\cdot\vec{\nabla})_{b_{2s} b'_{2s}}\, D(x-y) \tag{VI.8}$$

where $D(x-y)$ is the Jordan Pauli non massive function, κ is a constant to be determined, $\mathcal{P}$ denotes all possible permutations among the spinor indices and where we use convention (VI.2).

Then we expand the two component φ fields in term of the new field $\vec{f}$ (IV.17); imposing (VI.8) we obtain commutation relations for the $\vec{f}$ fields.

In the particular case of a spin 1 particle where (IV.19)

$$f_\mu(x) = \frac{1}{2} \epsilon_{0\mu\alpha\beta} \mathcal{F}^{\alpha\beta}(x)$$

postulating the plane wave expansion of $A^\mu(x)$

$$A^\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p^0}} \sum_{\lambda=1}^{2} \varepsilon^\mu(p,\lambda) \left\{ A(p,\lambda)\, e^{-ipx} + A^+(p,\lambda)\, e^{ipx} \right\} \tag{VI.9}$$

where
$$p = (p^0, \vec{p})$$
$$[A(p,\lambda), A^+(p',\lambda')] = \delta^3(\vec{p} - \vec{p}')\delta_{\lambda\lambda'}$$

λ and λ' are polarizations and using (IV.19) we obtain one more expression for the commutator between f fields.

We obtain so following expression for sum over polarization for a massless particle of spin 1, momentum p, massless:

$$\tilde{X}^{\mu\nu}(p) = \sum_{\lambda=1}^{2} \tilde{\varepsilon}^{\mu}(p,\lambda)\, \tilde{\varepsilon}^{\nu *}(p,\lambda) = -g^{\mu\nu} + \eta^{\mu}\eta^{\nu} - \hat{p}^{\mu}\hat{p}^{\nu} \qquad \text{(VI.10)}$$

where we have adopt Bjorken's notation

$$\eta^{\mu} = (1,0,0,0)$$

$$\hat{p}^{\mu} = \frac{\sqrt{(p\cdot\eta)^2 - p^2}}{p^{\mu} - (p\cdot\eta)\eta^{\mu}} .$$

Feynman rules

We take $\mathcal{L}_{\rm int}$ (III.13) with $g_1 = 0$ as a starting point

$$\mathcal{L}_{\rm int} = g_2\, \bar{\psi}_{a_1 a_2}\, \eta_{a_1}\, \eta'_{a_2} = g_2\, {}^t\eta\, \bar{\psi}\, \eta'$$

We assume that ψ corresponds to a particle of four momentum p_1 and spin 1, η has four momentum p_2 and η' has four momentum p_3 and spin $\frac{1}{2}$.

So we obtain the following result

$$-i\sqrt{m}\left\{(C^{-1}\gamma^{\mu})_{a_1 a_2} - \frac{i}{m}(p_1)_{\nu}\,(C^{-1}\sigma^{\mu\nu})_{a_1 a_2}\right\} . \qquad \text{(VI.11)}$$

Consider now the 3 below cases:

1°) η and η' are positive energy particles, so

$${}^t\eta = {}^t u = -\bar{v}\,C$$
$$\eta' = u'$$

the vertex above takes the form

$$i\sqrt{m}\,\bar{v}(p_2)\left\{\gamma^{\mu} - \frac{i}{m}(p_1)_{\nu}\,\sigma^{\mu\nu}\right\} u'(p_3) .$$

2°) η and η' are negative energy particles, so

$${}^t\eta = {}^t v = -\bar{u}\,C$$
$$\eta' = v'$$

the vertex takes the form

$$i\sqrt{m}\,\bar{u}(p_2)\left\{\gamma^{\mu} - \frac{i}{m}(p_1)_{\nu}\,\sigma^{\mu\nu}\right\} v'(p_3) .$$

3°) η is a positive energy particle and η' is a negative energy particle

$${}^t\eta = {}^t u = -\bar{v}\,C$$
$$\eta' = v'$$

the vertex takes the form

$$i\sqrt{m}\,\bar{v}(p_2)\left\{\gamma^{\mu} - \frac{i}{m}(p_1)_{\nu}\,\sigma^{\mu\nu}\right\} v'(p_3) .$$

Appendix

– The metric used is $g_{\mu\nu} = (1,-1,-1,-1)$

– Dirac's matrices commutation rules

$$\frac{1}{2}[\gamma^\mu,\gamma^\nu] = g^{\mu\nu}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] \quad ; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma^\mu = (\gamma^0,\vec{\gamma})$$

$$\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$$

$$\vec{\Sigma} = \gamma^5\,\gamma^0\,\vec{\gamma}$$

– In chiral representation

$$\gamma^5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \vec{\Sigma} = \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix} \qquad \gamma^\mu_{\alpha\beta} = \begin{bmatrix} 0 & -(\sigma_\mu)_{\alpha\beta} \\ -(\sigma^\mu)^{\alpha\beta} & 0 \end{bmatrix}$$

– Charge conjugation matrix $(C)_{\alpha\beta} = \begin{bmatrix} -i\sigma^2_{\alpha\beta} = C_1 & 0 \\ 0 & i\sigma^2_{\alpha\beta} = C_1 \end{bmatrix}$

– Pauli matrices are define as below by

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

By convention $\sigma^\mu = (\sigma^0 = 1, \vec{\sigma})$ designate $(\sigma^\mu)_{\alpha\dot\beta}$.

We have used also the property that:

$$(\sigma^0)_{\alpha\dot\beta} = (\sigma^0)_{\alpha\dot\beta} \qquad (\sigma^k)_{\alpha\dot\beta} = -(\sigma_k)_{\alpha\dot\beta} .$$

C_1 denotes the charge conjugation matrix in (2,2) space and obeys ${}^tC_1 = -C_1, C_1^\dagger = C_1^{-1}$.

VI – CONCLUSIONS

In this paper we have extended, to massless particles of arbitrary spin, the usual spin $\frac{1}{2}$ chiral components. The extended chiral components are eigenstates of generalized helicity operators. In a particle of spin s the eigenvalues of the helecity states are, as expected, in the range $[-s, s]$. The chiral components associated to the $-s$ and $+s$ eigenvalues satisfy Weyl's equations. The other components satisfy equations, in terms of two components spinors, analogous to Weyl's.

We have made a systematic analysis of the spin 1 particles. In this case there are four chiral components ψ_{RR}, ψ_{LL}, ψ_{RL} and ψ_{LR}. The helicity eigenvalue +1 (–1) is associated to ψ_{RR} (ψ_{LL}) whereas the 0 eigenvalue is associated to the components ψ_{RL}, ψ_{LR}. These components, on the other hand, have a very simple meaning in this case. They are associated to observable fields (like the electric and magnetic fields) or to potentials. That is

ψ_{RR} ψ_{LL} $\longleftrightarrow$ observable fields (E or H)
(Gauge invariant)

ψ_{LR} ψ_{RL} $\longleftrightarrow$ potentials (A_μ)
(not Gauge invariant)

We have shown that all chiral components are, in fact, important dynamical variables for getting a complete description of electrodynamics. The equations associated to some components are just Maxwell's equations whereas other equations give constraints in the potentials, that, in this case, is Lorentz condition.

One of the advantages of this approach is that it allows to formulate electrodynamics in terms of potentials, by using two components spinors, or in terms of the observable fields (Maxwell equation) E and H.

The next question that we asked ourselves is if ordinary matter prefers, or not, to couple with some chiral components. If matter does not distinguish between chiral components, then it should couple with the field ψ. That is not the case.

On the other hand, if ordinary matter couples only with the ψ_{RR} ψ_{LL} we would have electrodynamics formulated entirely in terms of observables. That is not the alternative that nature chooses either.

The usual QED is compatible only with a theory in which only ψ_{LR} (ψ_{RL}) couples with the ordinary matter. From this point of view QED is another example of an asymmetric interaction between chiral components.

The coupling of ordinary matter with some chiral components, in the case of the photon, does not have any consequence like parity violation. It is just a dynamical consequence that in this case is the minimal coupling.

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REFERENCES

1) R.P. Feynman and M. Gell–Mann, *Phys. Rev.* **109**, 193 (1958); S.S. Gerstein and Y.B. Zeldovich, *Sov. Phys. JETP* **2**, 576 (1956); E.C.G. Sudarshan and R.E. Marshak, *Phys. Rev.* **109**, 1860 (1958); J.J. Sakurai, *Nuovo Cimento* **7**, 649 (1958).

2) T.D. Lee and C.N. Yang, *Phys. Rev.* **104**, 254 (1956); *Phys. Rev.* **105**, 1671 (1957); C.S. Wu et al., *Phys. Rev.* **105**, 1413.

3) V. Bargmann and E.P. Wigner, *Proc. Nat. Acd. Sci.* **34**, 211 (1948).

4) S.D. Drell and J.D. Bjorken, *Relativistic Quantum Mechanics*, New York, McGraw–Hill Book Co. (1964).

5) D. Spehler, *Effective interactions involving massive and massless particles of spin* $\frac{1}{2}$, 1, $\frac{3}{2}$ *and* 2, to appear in *Journal of Math. Physics.*

6) D. Spehler, PhD Thesis, *Sur la theorie des champs a spin* 1 *et* $\frac{3}{2}$, preprint CRN/HE 79–10.