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INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
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M. Nielsen
Instituto de Física, Universidade de São Paulo

J. da Providência
Centro de Física Teórica (INIC), Universidade de Coimbra,
P-3000 Coimbra, Portugal

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M. NIELSEN

Instituto de Física, Universidade de São Paulo Caixa Postal 20516 - 01498 São Paulo, Brazil

J. da PROVIDÊNCIA

Centro de Física Teórica (INIC), Universidade de Coimbra, P–3000 Coimbra, Portugal

Abstract: Using a boson expansion that includes appropriate anharmonic terms, the decay width of a scalar-isoscalar meson into two pseudoscalar-isovector mesons is calculated. A confining mechanism, leading to a discretization of the modes in the quark-antiquark continuum, is incorporated in the model through a constrained RPA approach. The technique is applied to the bound state solution as well as to the discretized solution.

It is a well established empirical fact that the properties and interactions of non-strange hadrons at energy and momentum tranfers up to about 1GeV are most efficiently described in terms of mesons, and in particular, pion degrees of freedom rather than quark-gluon degrees of freedom.

As a consequence, much activity has been focussed on the derivation of effective Lagrangians for low energy hadron physics [1-3]. The basic starting point in the construction of a low energy effective Lagrangian of QCD is the chiral symmetry [4]. The importance of chiral invariance is realized by noticing that it ensures the appearance of Goldstone bosons, and the existence of current algebra relations and of low-energy theorems.

The linear σ model originally suggested by Gell - Mann and Lévy [5] allows for spontaneously broken chiral symmetry and involves, besides quarks, a sigma field (scalar-isoscalar) and a pion field (pseudoscalar-isovector). By a slight explicit breaking of the chiral symmetry one can fulfil the hypothesis of the partial conservation of the axial current [5,6].

The aim of the present paper is to calculate the decay width of the scalar-isoscalar mesons into two pions within the framework of the linear sigma model. To do this we will consider a bosonization technique, which takes into account anharmonic terms responsible for the coupling between mesons. We will restrict ourselves to the time dependent Hartree-Fock (TDHF) approach in the small amplitude limit of the meson field description. When we consider the effects of the generator up to second order, we get linearized equations of motion for the excitation modes, equivalente to Random Phase Approximation (RPA). An extension of this expansion up to third order will include anharmonic terms associated with the $\sigma\pi\pi$ coupling.

In the framework of the TDHF formalism the time evolution of Slater determinants describing systems in interaction with external fields σ, $\vec{\phi}$, is governed by the Lagrangian

$$L = i\,tr(\dot{U}\rho_0 U^+) + \int d^3x(\dot{\sigma}\Pi_\sigma + \dot{\vec{\phi}}.\vec{\Pi}_\phi) - E(\rho,\sigma,\Pi_\sigma,\vec{\phi},\vec{\Pi}_\phi)\,, \qquad (1)$$

where ρ_0 is the one-body density matrix, equivalente to the Slater determinant which minimizes the energy of the system. Here U is an arbitrary unitary matrix that provides a density matrix displaced from equilibrium

$$\rho = U\rho_0 U^+\,, \qquad (2)$$

and

$$E(\rho,\sigma,\Pi_\sigma,\vec{\phi},\vec{\Pi}_\phi) = tr(h\rho) + E(\sigma,\Pi_\sigma,\vec{\phi},\vec{\Pi}_\phi)\,,$$

where $E(\sigma,\Pi_\sigma,\vec{\phi},\vec{\Pi}_\phi)$ is the energy of the free meson fields, and h is the one body Hamiltonian.

If we parametrize the matrix U and the fields $\sigma,\Pi_\sigma,\vec{\phi},\vec{\Pi}_\phi$ in terms of the RPA operators (θ_n,θ_n^+) and of the RPA fields $(\delta\sigma_{\pm n},\Pi_{\sigma\pm n},\delta\vec{\phi}_{\pm n},\vec{\Pi}_{\phi\pm n}$, the subscripts $\pm$ standing for positive and negative frequency solutions) i.e. :

$$U = exp(i\sum_n(c_n\theta_n^+ + c_n^*\theta_n)) = exp(iS)\,, \qquad (3.a)$$

$$\begin{pmatrix} \sigma - <\sigma> \\ \Pi_\sigma \\ \vec{\phi} - <\vec{\phi}> \\ \vec{\Pi}_\phi \end{pmatrix} = \sum_n c_n \begin{pmatrix} \delta\sigma_{+n} \\ \Pi_{\sigma+n} \\ \delta\vec{\phi}_{+n} \\ \vec{\Pi}_{\phi+n} \end{pmatrix} + \sum_n c_n^* \begin{pmatrix} \delta\sigma_{-n} \\ \Pi_{\sigma-n} \\ \delta\vec{\phi}_{-n} \\ \vec{\Pi}_{\phi-n} \end{pmatrix} , \tag{3.b}$$

where $<\sigma>$ and $<\vec{\phi}>$ denote the equilibrium fields, the Lagrangian becomes

$$L = \frac{i}{2}\sum_n (\dot{c}_n^* c_n - c_n^* \dot{c}_n) - \mathcal{H}(c, c^*) . \tag{4}$$

The mean-field Hamiltonian of the system, $\mathcal{H}(c, c^*)$, is nothing else than the energy $E(\rho, \sigma, \Pi_\sigma, \vec{\phi}, \vec{\Pi}_\phi)$ expressed in terms of the parameters c.

It is well known that anaharmonic processes are responsible for the decay of collective excitations associated with normal modes of many-body systems [7]. Therefore, the description of physical mesons as collective excitations of the vacuum, which is here regarded as a many-body system, automatically implies the interpretation of mesonic decays as such anharmonic processes. This philosophy has been followed in ref. [8] in connection with the Nambu-Jona-Lasinio model and is taken again here in connection with the sigma model. In the small amplitude limit the Hamiltonian becomes diagonal (harmonic approximation). If the expansion is extended beyond the harmonic order we get:

$$\mathcal{H}(c, c^*) = \sum_n \omega_n c_n c_n^* + \sum_{l,m,n} (k_{lmn} c_l^* c_m^* c_n^* +$$
$$+ h_{lmn} c_l^* c_m^* c_n + k_{lmn}^* c_l c_m c_n + h_{lmn}^* c_l c_m c_n^*) + \dots \tag{5}$$

The term $h_{lmn}^* c_l c_m c_n^*$ describes the decay of a boson n into a boson l and a boson m.

To study the decay of a meson σ into two π we have to evaluate $h_{\pi\pi\sigma}^*$. The correspondig coupling, $g_{\sigma\pi\pi}$, is by definition

$$h_{\pi\pi\sigma}^* = \frac{g_{\sigma\pi\pi}}{2\omega_\pi \sqrt{2\omega_\sigma \Omega}} , \tag{6}$$

where Ω is the normalization volume.

Using the Fermi golden rule, the transition amplitude for the decay $\sigma \to \pi\pi$, in the chiral limit, is

$$\Gamma_{\sigma\pi\pi} = \frac{3}{16\pi\omega_\sigma} |g_{\sigma\pi\pi}|^2 . \tag{7}$$

Here the factor 3 stands for the isospin degeneracy.

In a classical realization of the linear sigma model, the scalar-isoscalar field σ and the pseuscalar-isovector field $\vec{\phi}$ are classical fields. The effective Hamiltonian of the system can be written as

$$H = \int d^3x \overline{\Psi}(x)(\vec{p}.\vec{\gamma} + g(\sigma(x) + i\gamma_5 \vec{\tau}.\vec{\phi}(x)))\Psi(x)$$
$$+ \frac{1}{2}\int d^3x \,(\Pi_\sigma^2 + \vec{\nabla}\sigma \cdot \vec{\nabla}\sigma + \Pi_\phi^2 + \vec{\nabla}\phi_i \cdot \vec{\nabla}\phi_i) + \int d^3x \left(\frac{g^2}{2}(\sigma^2 + \phi^2 - \sigma_0^2)^2 - c\sigma\right)$$

$$+ 2\xi \int \frac{d^3p d^3x}{(2\pi)^3} \sqrt{p^2 + g^2(\sigma^2 + \phi^2)}\, \Theta(\Lambda^2 - p^2) , \tag{8}$$

where $\vec{\gamma} = \beta\vec{\alpha}$ and γ_5 are the usual Dirac matrices, $\vec{\tau}$ corresponds to the matrices of the fundamental flavor representation SU(2), Π_σ and $\vec{\Pi}_\phi$ are the conjugate momentum associated with the classical fields σ and $\vec{\phi}$ respectively . The last term in eq.(8) is a renormalization term, which depends on a cutoff parameter Λ.The factor ξ in this term stands for the degeneracy of the system and will be taken equal to six (three colours and two flavours). This counterterm cancels the entire one fermion-loop corrections, but the inclusion of the Dirac sea plays an important role in insuring the stability of the vacuum against fluctuations [9]. The constants c and σ_0 in eq.(8) are given by

$$c = m_\pi^2 f_\pi \tag{9.a}$$

$$g^2\sigma_0^2 = M^2 - \frac{m_\pi^2}{2} . \tag{9.b}$$

where f_π is the pion decay constant, $M = g <\sigma>_{vac}$ is the constituent quark mass and m_π is the mass of the phenomenological structureless pion related to the pseudoscalar-isovector field. With this choice of these constants the Goldberger-Treiman relation to this order : $f_\pi = M_0/g$, is fulfilled.

In terms of the generator S of the vacuum fluctuations, corresponding to a sum of generic modes of excitation (see eq.(3)) the functional of energy $E(\rho, \sigma, \Pi_\sigma, \vec{\phi}, \vec{\Pi}_\phi)$ expanded up to third order, in homogeneous quark matter, is

$$E(\rho, \sigma, \Pi_\sigma, \vec{\phi}, \vec{\Pi}_\phi) = E_0 + \frac{1}{2}tr(\rho_0[S, [h_0, S]]) + itr(\rho_0[\delta h, S]) - \frac{\Omega}{2}\left(\Pi_\sigma^2 + (\vec{\Pi}_\phi)^2\right)$$
$$+ \Omega(2M^2(\delta\sigma)^2 + \frac{m_\pi^2}{2}(\delta\vec{\phi})^2) + \xi g^2 \Omega \int \frac{d^3p}{(2\pi)^3}\left(\frac{p^2}{\epsilon^3}(\delta\sigma)^2 + \frac{1}{\epsilon}(\delta\vec{\phi})^2\right)\Theta(\Lambda^2 - p^2)$$
$$+ \frac{i}{3!}tr(\rho_0[S, [S, [S, h_0]]]) + \frac{1}{2}tr(\rho_0[S, [\delta h, S]]) + 2\Omega g M((\delta\sigma)^3 + (\delta\vec{\phi})^2\delta\sigma)$$
$$- \xi g^3 M \Omega \int \frac{d^3p}{(2\pi)^3}\left(\frac{p^2}{\epsilon^5}(\delta\sigma)^3 + \frac{(\delta\vec{\phi})^2\delta\sigma}{\epsilon^3}\right)\Theta(\Lambda^2 - p^2) , \tag{10}$$

where $E_0 = tr(h_0\rho_0) + E(<\sigma>, <\vec{\phi}>)$, $h_0 = \vec{p}.\vec{\alpha} + \beta M$,

$$\rho_0 = \frac{1}{2}\left(I - \frac{h_0}{\epsilon}\right)\Theta(\Lambda^2 - p^2) ,$$

is the one-body density matrix that describes the ground state of the system (the vacuum), $\epsilon = \sqrt{p^2 + M^2}$ and

$$\delta h = \beta g \delta\sigma + i g \beta\gamma_5 \vec{\tau}.\delta\vec{\phi} , .$$

The generators of the scalar-isoscalar and pseudoscalar-isovector homogeneous excitations for zero momentum transfer are respectively given by

$$S_\sigma = \vec{p}.\vec{\alpha} F_1(p^2, t) + i\beta\vec{p}.\vec{\alpha}\, F_2(p^2, t) , \tag{11.a}$$

$$S_\pi = i\beta\gamma_5\vec{\tau}.\vec{S}_1(p^2,t) + \gamma_5\vec{\tau}.\vec{S}_2(p^2,t)\,. \tag{11.b}$$

When the RPA equations for these generators are exactly solved [10] we obtain two types of solutions: 1) the discrete bound meson states, which have masses ω smaler than the sum of the dynamical masses of their quark-antiquark ($q\bar{q}$) constituents ($\omega \leq 2M$), and 2) the quasi-bound states embedded in the continuum which have masses larger than $2M$. A technique for discretization of the continuum solutions may be introduced in the framework of a convenient variational approach. This method, which may be regarded as a simulation of confinement, since it represents a constrain imposed on the relative motion of the $q\bar{q}$ pair, relies on the assumption that the generators of the vacuum fluctuations do not depend on p, except for the factor $\vec{p}\cdot\vec{\alpha}$ occuring in S_σ. This assumption is not an approximation to avoid the exact RPA treatment but is rather an ansatz aimed at simulating a confining mechanism. It implies that the hadronic radius is proportional to the inverse of the cutoff parameter Λ. The assumption,

$$F_i(p^2,t) = f_i(t), \tag{12.a}$$

$$\vec{S}_i(p^2,t) = \vec{s}_i(t), \tag{12.b}$$

leads to the replacement of the the exact continuum RPA solutions by a single normalizable mode, that represents a kind of confined mode. The constrained RPA allows to describe the mesons above $2M$, without changing the main features which are connected with chiral invariance. In ref. [11] a similar assumption has been considered in connection with the Nambu-Jona-Lasinio Model.

The orthonormalized RPA solutions for both, scalar-isoscalar and pseudoscalar- isovector modes, when the assumption given by eq.(12) is considered, are given by :

a)scalar-isoscalar

$$|\delta\sigma_{\pm n}| = \frac{1}{\sqrt{2|\omega_{\sigma n}|\Omega}}\frac{1}{\sqrt{1+\frac{8\xi g^2 I_2^3}{(\omega_{\sigma n}^2 I_2 - 4I_{\epsilon 2})^2}}},$$

$$\Pi_{\sigma\pm n} = \pm i\omega_{\sigma n}\delta\sigma_{\pm n},$$

$$f_{1\pm n} = \mp\frac{i\omega_{\sigma n}}{M}\frac{gI_2}{\omega_{\sigma n}^2 I_2 - 4I_{\epsilon 2}}\delta\sigma_{\pm n},$$

$$f_{2\pm n} = \frac{2gI_2}{\omega_{\sigma n}^2 I_2 - 4I_{\epsilon 2}}\delta\sigma_{\pm n}\,. \tag{13.a}$$

The dispersion relation is

$$\omega_{\sigma n}^2 = \frac{2I_{\epsilon 2}}{I_2} + 2M^2 + \xi g^2 I_2' \pm \sqrt{\left(\frac{2I_{\epsilon 2}}{I_2} - 2M^2 - \xi g^2 I_2'\right)^2 + 8\xi g^2 I_2}\,, \tag{14.a}$$

where $n = 1,2$.

b)pseudoscalar-isovector

$$|\delta\vec{\phi}_{\pm n}| = \frac{1}{\sqrt{2|\omega_{\pi n}|\Omega}}\frac{1}{\sqrt{1+\frac{8\xi g^2 I_0^2 I_{\epsilon 0}}{(\omega_{\pi n}^2 I_0 - 4I_{\epsilon 0})^2}}}$$

$$\Pi^j_{\phi\pm n} = \pm i\omega_{\pi n}\delta\phi^j_{\pm n}$$

$$s^j_{1\pm n} = \pm i\omega_{\pi n}\frac{gI_0}{\omega_{\pi n}^2 I_0 - 4I_{\epsilon 0}}\delta\phi^j_{\pm n}$$

$$s^j_{2\pm n} = \frac{2gI_{\epsilon 0}}{M(\omega_{\pi n}^2 I_0 - 4I_{\epsilon 0})}\delta\phi^j_{\pm n}\,, \tag{13.b}$$

and the dispersion relation is

$$\omega_{\pi n}^2 = \frac{2I_{\epsilon 0}}{I_0} + \frac{m_\pi^2 + 2\xi g^2 I_0}{2} \pm \sqrt{\left(\frac{2I_{\epsilon 0}}{I_0} - \frac{m_\pi^2 + 2\xi g^2 I_0}{2}\right)^2 - 4m_\pi^2\frac{I_{\epsilon 0}}{I_0}}\,. \tag{14.b}$$

In eqs.(13) and (14) the symbols I_n, I_n', and $I_{\epsilon n}$, $n = 0,2$ stand for the following integrals

$$I_n = \int\frac{d^3p}{(2\pi)^3}\frac{p^n}{\epsilon}\Theta(\Lambda^2 - p^2),$$

$$I_n' = \int\frac{d^3p}{(2\pi)^3}\frac{p^n}{\epsilon^3}\Theta(\Lambda^2 - p^2),$$

$$I_{\epsilon n} = \int\frac{d^3p}{(2\pi)^3}p^n\epsilon\,\Theta(\Lambda^2 - p^2)\,.$$

With these complete sets of solutions we can construct the RPA operators (eq.(3)) as :

$$\theta^+_{\sigma n} = \vec{p}.\vec{\alpha}f_{1+n} + i\beta\vec{p}.\vec{\alpha}f_{2+n}\,, \tag{15.a}$$

$$\theta^+_{\pi n} = i\beta\gamma_5\tau^j s^j_{1+n} + \gamma_5\tau^j s^j_{2+n}\,. \tag{15.b}$$

From this we get

$$S = \sum_{\substack{r=\sigma,\pi \\ n=1,2}} (c_{rn}\theta^+_{rn} + c^*_{rn}\theta_{rn})\,. \tag{15.c}$$

We can also expand any arbitrary oscillation of the scalar-isoscalar and pseudoscalar-isovector fields, in terms of the RPA solutions. It gives:

$$\delta\sigma = \sum_{n=1}^{2}(c_{\sigma n}\delta\sigma_{+n} + c^*_{\sigma n}\delta\sigma_{-n})\,, \tag{16.a}$$

$$\delta\phi^i = \sum_{n=1}^{2}(c_{\pi n}\delta\phi^i_{+n} + c^*_{\pi n}\delta\phi^i_{-n})\,. \tag{16.b}$$

Using eqs.(15) and (16) in eq.(10), and identifying the coefficient of $c_{\pi 1}c_{\pi 1}c^*_{\sigma n}$ as $h^*_{\sigma\pi\pi}$ (that means that the pion is taken as the RPA bound-state solution), we get

$$h^*_{\sigma\pi\pi} = Mg\Omega(2 - \xi g^2 I_0')|\delta\vec{\phi}_{+1}|^2\delta\sigma_{-n} - 4\xi g\Omega(I_2\delta\phi^j_{+1}(s^j_{1+1}f_{1-n} +$$
$$+ s^j_{2+1}f_{2-n}) - MI_0((\vec{s}_{1+1})^2 + (\vec{s}_{2+1})^2)\delta\sigma_{-n})\,. \tag{17}$$

From eqs.(6) and (7) we can evaluate the $\sigma \rightarrow \pi\pi$ coupling constant and the width for the decay of a sigma into two pions for both, the bound-state solution and the discretized solution, given by eq.(14.a).

In table I, the numerical values of $\omega_\sigma, g_{\sigma\pi\pi}$ and $\Gamma_{\sigma\pi\pi}$, are shown . In order to reproduce the experimental pion decay constant, $f_\pi = 93 MeV$, for a constituent quark mass $M = 320 MeV$ we get from the Goldberger-Treiman relation $g = 3.44$. If we want the bound-state solution of eq.(14.b) to reproduce the physical pion mass : $\omega_{\pi 1} = 138 MeV$, and the discretized solution to reproduce the pion ressonance, $\omega_{\pi 2} = 1300 MeV$, we get $m_\pi = 179 MeV$ and $\Lambda = 1.62 M$. These are the values of the parameters used to derive the numerical results presented in table I. [6 Our results are lower than the σ model result : $g_{\sigma\pi\pi} = \frac{\omega_\sigma^2}{2 f_\pi}$, in agreement with results obtained by others authors [8,12] in the NJL model. This is expected since the σ model result only gives the coupling between the phenomenological structureless sigma and pion fields. In our approach the mesons must be understood as composed of two ingredients : the structureless field plus a collective $q\bar{q}$ component built on the non-perturbative vacuum. Therefore the coupling constant between these mesons must be different from the coupling between the fields.

The two lowest established scalar-isoscalar resonances are the $f_0(975)$ and $f_0(1400)$ which decay widths are respectively $\Gamma = 34 MeV$ ($\approx 78\%$ into $\pi\pi$ and $\approx 23\%$ into $K\overline{K}$) and $\Gamma = 150 - 400 MeV$ ($\approx 94\%$ into $\pi\pi$ and $\approx 6\%$ into $K\overline{K}$) [13]. The numerical results provide support to compare the second mode to the $f_0(1400)$, which is more commonly interpreted as a true resonance [13,14]. The decay width obtained is in good agreement with the experimental one.

In conclusion, we have used the linear sigma model to investigate the decay width of a sigma meson into two pions. In this approach mesons are composed by phenomenological structureless fields plus excitations of the vacuum. They are described as RPA solutions of the linearized TDHF equations. Decay processes of mesons are described by anharmonic contributions of the vacuum fluctuations energy. The problems of dealing with the RPA modes in the continuum, associated with the lack of confinement, are circumvented by imposing a simple restriction on the generators of the vacuum fuctuations. The discretized modes of the continuum can decay in specific meson channels. Despite the fact that the present results should be regarded as essentially qualitative, the decay width obtained is of the same order of the experimental one.

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TABLE CAPTION

Table I- Values of the scalar-isoscalar meson mass (ω_σ), sigma-pion-pion coupling constant ($g_{\sigma\pi\pi}$) and the decay width of a sigma into two pions ($\Gamma_{\sigma\pi\pi}$), for bouns-states (first line) and discretized solution, calculated for the pion RPA bound-state solution using $M = 320 MeV$, $g = 3.44$, $m_\pi = 179 MeV$ and $\Lambda = 1.62M$.

TABLE I

$\omega_\sigma(MeV)$	$g_{\sigma\pi\pi}(MeV)$	$\Gamma_{\sigma\pi\pi}(MeV)$
550.	780.	67.0
1280.	3150.	470.