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PUBLICAÇÕES

IFUSP/P-912

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Maio/1991

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ABSTRACT

We derive a formally exact normal ordered boson expansion in terms of harmonic oscillator bosons, for the nuclear charge and current densities. This expansion separates the dependence on the momentum transfer, from the dependence on the nuclear structure, and is exact in the case of the harmonic oscillator shell model space. Our boson expansion provides a mechanism for expressing the multipole operators in terms of SU(3) tensors. Thus, we perform an SU(3) tensor analysis of the multipole operators, since this is of major importance for shell–model calculations in an SU(3) basis.

1. Introduction

In references 1 and 2 it was shown that the charge and current density multipoles, when restricted to a harmonic oscillator shell–model space, can be written as a gaussian times a polynomial in the momentum transfer, q, where the coefficients of the powers of q are specific one–body operators. In turn, these one–body operators are expressed as a normal ordered number–conserving polynomial in terms of the oscillator boson creation and annihilation operators.

A consequence of this result is that different shell–model determinations of the form factors must differ from each other only in the values of a few parameters, defined as the matrix elements of the one–body operators. Thus, for example, it was found that in the sd–shell, the shape of the longitudinal $F^{L}_{0^+\to 4^+}(q)$ and transverse $F^{T}_{0^+\to 4^+}(q)$ and $F^{T}_{0^+\to 5^+}(q)$ form factors are independent of the nuclear wave functions, all harmonic oscillator sd–shell model calculations giving identical shapes (and this includes calculations for different nuclei and different states in a given nucleus)[2].

Although all the discussion in references 1 and 2 was focused in the sd–shell this boson expansion has a much greater significance and generality.

In this paper we will deduce a formally exact boson expansion of the charge and current density multipoles. The general form of the expansion is analogous to the one in reference 2, only now the series in powers of q is not limited and the one–body operators have boson number conserving and boson number non–conserving terms. Noticing that, for states restricted to the sd–shell, we can annihilate at most two oscillator quanta per particle and only the boson number conserving terms contribute, the series expansion terminates as in reference 2. However even in the general case the boson expansion should be rapidly convergent as can be seen by noticing that it reduces to a polynomial with a finite number of terms when we restrict the many body Hilbert–space to a space constructed with a finite set of harmonic oscillator single particle wave functions.

The boson expansion that we employ can be regarded as an application of second quantization techniques. We recall that the shell–model makes substancial use of fermion second quantization. In particular, it uses the fact that any one–body operator can be expanded in a basis of elementary one–body operators with the result that the matrix elements of one–body operators can be expressed as known linear combinations of the matrix elements of the basis operators. References 1 and 2 and the present paper not only show that similar results are obtained with boson expansions but that there are very substancial practical advantages to this alternative approach.

One advantage is that there are only three different harmonic oscillator bosons per nucleon, as opposed to an infinite number of single–fermion states. Therefore the boson expansion can be much more economical and, as ilustrated in ref. 2 for the case of the harmonic oscillator shell–model, the matrix elements of the charge and current density multipoles depend on a smaller set of independently basic matrix elements than fermion second quantization might lead one to suppose. Admitedly, only a small number of fermion states contribute on restriction to a single h.o. shell. However, the finite fermion expansion relevant for one h.o. shell have nothing to say about the corresponding expansions for a different h.o. shell. In contrast a boson expansion is simultaneously valid for all nuclei and all h.o. shells. For example, by considering the action of an arbitrary one–body operator in nuclear states with one nucleon in the h.o. sd–shell one can deduce all the one–fermion matrix elements needed for application to any other h.o. sd–shell nucleus. But, in a similar way, one can deduce all the one and two–boson matrix elements needed for application to any nucleus, restricted to a h.o. shell.

A second motivation for considering the boson expansion of operators is because of the importance of microscopic nuclear collective models that make use of the boson structure of the harmonic oscillator shell–model[3]. For example, it has recently been pointed out that enormous benefits may result from the symplectic shell model, in which

calculations are carried out in an LS coupled basis that reduces both the symplectic group $S_p(3,R)$ and its SU(3) subgroup.[3] Such a basis provides a natural interpretation of shell–model wave functions in collective model terms and is expected to give more realistic results for nuclei in which deformation correlations predominate. The importance of the boson expansion arises from the fact that the boson creation and annihilation operators are respectively [10] and [01] SU(3) tensors. Therefore symmetrical polynomials of boson operators are readily coupled to irreducible SU(3) tensor operators, a property of considerable practical importance for calculations with shell–model wave functions in an SU(3) basis.

In this paper we will deduce a formally exact boson expansion for the charge and current density multipoles. In practice only a relatively small number of terms in the expansions should be relevant. For example, the low q behaviour of the lowest order charge and current density multipoles *for all nuclei* is dominated by terms up to quartic in a normal order expansion in terms of the harmonic oscillator boson creation and annihilation operators.

This paper is organized as follows. In section II we write expressions of the charge and current density multipoles which facilitates our task of finding its boson expansion. The boson expansion of the charge density multipoles is deduced in section III and the ones of the current density multipoles in section IV. The SU(3) tensor analysis of the charge and current densities multipoles is performed in section V. Our concluding remarks are presented in section VI.

II. Charge and current density multipoles

II.1. Charge density multipoles

The Fourier transform of the charge density operator, $\hat{\rho}(\underset{\sim}{x})$,

$$\hat{\rho}(\underset{\sim}{x}) = \sum_n \frac{(1 + \hat{\tau}_3(n))}{2} \delta(\underset{\sim}{x} - \hat{\underset{\sim}{r}}(n)) \quad , \tag{2.1}$$

is equal to

$$\hat{\rho}(\underset{\sim}{q}) = \int e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}} \hat{\rho}(\underset{\sim}{x})\, d^3\underset{\sim}{x} =$$

$$= \sum_n \frac{(1 + \hat{\tau}_3(n))}{2}\, e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)} \quad . \tag{2.2}$$

This operator can be expanded as

$$\hat{\rho}(\underset{\sim}{q}) = \sum_{LM} 4\pi\,(-i)^L\, Y^*_{LM}(\Omega\,(\underset{\sim}{q}))\, \hat{\rho}_{LM}(q) \quad , \tag{2.3}$$

where $\hat{\rho}_{LM}(q)$ is the charge density multipole operator

$$\hat{\rho}_{LM}(q) = \int j_L(qx)\, Y_{LM}(\Omega\,(\underset{\sim}{x}))\, \hat{\rho}(\underset{\sim}{x})\, d^3\underset{\sim}{x} \quad . \tag{2.4}$$

The charge density multipole $\hat{\rho}_{LM}(q)$ is a one–body operator and can be written as

$$\hat{\rho}_{LM}(q) = \sum_n \frac{(1 + \hat{\tau}_3(n))}{2} \hat{\not{\rho}}_{LM}(n;q) \quad , \tag{2.5}$$

where we see that the operator $\hat{\not{\rho}}_{LM}(n;q)$ appears in the multipole expansion of $e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)}$

$$e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)} = \sum_{LM} 4\pi\,(-i)^L\, Y^*_{LM}(\Omega\,(\underset{\sim}{q}))\, \hat{\not{\rho}}_{LM}(n;q) \quad . \tag{2.6}$$

As will be shown below, with the help of eqs.(2.5) and (2.6), once we find the boson expansion of the charge density multipoles, the boson expansion of the current density multipoles follow immediately.

II.2. Current density multipoles

The Fourier transform of the current density operator, $\hat{\underset{\sim}{j}}(\underset{\sim}{x})$,

$$\hat{\underset{\sim}{j}}(\underset{\sim}{q}) = \int e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}} \hat{\underset{\sim}{j}}(\underset{\sim}{x})\, d^3\underset{\sim}{x} \quad , \tag{2.7}$$

can be expanded as

$$\hat{\underset{\sim}{j}}(\underset{\sim}{q}) = \sum_{JM} 4\pi\,(-i)^J \Big[\underset{\sim}{Y}^{(long)*}_{JM}(\Omega\,(\underset{\sim}{q}))\, \hat{j}^{(long)}_{JM}(q) + \underset{\sim}{Y}^{(el)*}_{JM}(\Omega\,(\underset{\sim}{q}))\, \hat{j}^{(el)}_{JM}(q) + \underset{\sim}{Y}^{(mag)*}_{JM}(\Omega\,(\underset{\sim}{q}))\, \hat{j}^{(mag)}_{JM}(q) \Big] \quad , \tag{2.8}$$

where the $\hat{j}^{(long)}_{JM}(q)$, $\hat{j}^{(el)}_{JM}(q)$ and $\hat{j}^{(mag)}_{JM}(q)$ are, respectively, the longitudinal, transverse electric and transverse magnetic multipoles of the current density operator,[4,5]

$$\hat{j}^{(long)}_{JM}(q) = \int \frac{i}{q} \underset{\sim}{\nabla}\, j_J(qx)\, Y_{JM}(\Omega\,(\underset{\sim}{x}))\cdot \hat{\underset{\sim}{j}}(x) d^3\underset{\sim}{x} \tag{2.9a}$$

$$\hat{j}^{(el)}_{JM}(q) = \frac{1}{q}\int \underset{\sim}{\nabla} \times j_J(qx)\, \underset{\sim}{Y}_{JJ;M}(\Omega(\underset{\sim}{x}))\cdot \hat{j}(\underset{\sim}{x})\, d^3\underset{\sim}{x} \tag{2.9b}$$

$$\hat{j}^{(mag)}_{JM}(q) = \int j_J(qx)\, \underset{\sim}{Y}_{JJ;M}(\Omega(\underset{\sim}{x}))\cdot \hat{j}(\underset{\sim}{x})\, d^3\underset{\sim}{x} \;, \tag{2.9c}$$

where $\underset{\sim}{Y}_{JJ;M}(\Omega(\underset{\sim}{x}))$ are the vector spherical harmonics (eq. 2.17) and $\underset{\sim}{Y}^{(long)}_{JM}(\Omega(\underset{\sim}{q}))$, $\underset{\sim}{Y}^{(el)}_{JM}(\Omega(\underset{\sim}{q}))$ and $\underset{\sim}{Y}^{(mag)}_{JM}(\Omega(\underset{\sim}{q}))$ are respectively, the longitudinal, transverse electric and transverse magnetic spherical vector fields[6])

$$\underset{\sim}{Y}^{(long)}_{JM}(\Omega(\underset{\sim}{q})) = \sqrt{\frac{2J+1}{4\pi}}\, D^{J*}_{M0}(\Omega(\underset{\sim}{q}))\, \underset{\sim}{\xi}_0(\Omega(\underset{\sim}{q})) = \frac{\underset{\sim}{q}}{q}\, Y_{JM}(\Omega(\underset{\sim}{q})) \tag{2.10a}$$

$$\underset{\sim}{Y}^{(el)}_{JM}(\Omega(\underset{\sim}{q})) = \sqrt{\frac{2J+1}{8\pi}}\left[D^{J*}_{M-1}(\Omega(\underset{\sim}{q}))\, \underset{\sim}{\xi}_{-1}(\Omega(\underset{\sim}{q})) + D^{J*}_{M1}(\Omega(\underset{\sim}{q}))\, \underset{\sim}{\xi}_{1}(\Omega(\underset{\sim}{q}))\right] \tag{2.10b}$$

$$\underset{\sim}{Y}^{(mag)}_{JM}(\Omega(\underset{\sim}{q})) = \sqrt{\frac{2J+1}{8\pi}}\left[D^{J*}_{M-1}(\Omega(\underset{\sim}{q}))\, \underset{\sim}{\xi}_{-1}(\Omega(\underset{\sim}{q})) - D^{J*}_{M1}(\Omega(\underset{\sim}{q}))\, \underset{\sim}{\xi}_{1}(\Omega(\underset{\sim}{q}))\right]. \tag{2.10c}$$

In eq.(2.10), the vectors $\underset{\sim}{\xi}_\mu(\Omega(\underset{\sim}{q}))$ are unit spherical basis vectors in a coordinate system whose z direction coincides with the direction of $\underset{\sim}{q}$ and $D^J_{MM'}(\Omega(\underset{\sim}{q}))$ is equal to,

$$D^J_{MM'}(\Omega(\underset{\sim}{q})) = D^J_{MM'}(\varphi_q, \theta_q, 0) \;,$$

where φ_q and θ_q are the azimuthal and polar angles, which determine the direction of $\underset{\sim}{q}$.

In what follows we will consider separately the convection and magnetization current densities[4,5]). In each case we will find expressions for the current multipoles which reduces the task of finding its boson expansion to the one of finding the boson expansion of the charge density multipoles.

II.3. Longitudinal, transverse electric and transverse magnetic multipoles of the convection current

The convection current, $\hat{\underset{\sim}{j}}^c(\underset{\sim}{x})$, and its Fourier transform, $\hat{\underset{\sim}{j}}^c(\underset{\sim}{q})$, are respectively equal to:

$$\hat{\underset{\sim}{j}}^c(\underset{\sim}{x}) = \frac{\hbar}{mc}\sum_n \frac{(1+\hat{\tau}_3(n))}{2}\frac{1}{2}\left[\frac{\hat{\underset{\sim}{p}}(n)}{\hbar}\,\delta(\underset{\sim}{x}-\hat{\underset{\sim}{r}}(n)) + \delta(\underset{\sim}{x}-\hat{\underset{\sim}{r}}(n))\,\frac{\hat{\underset{\sim}{p}}(n)}{\hbar}\right], \tag{2.11}$$

and

$$\hat{\underset{\sim}{j}}^c(\underset{\sim}{q}) = \frac{\hbar}{mc}\sum_n \frac{(1+\hat{\tau}_3(n))}{2}\frac{1}{2}\left[\frac{\hat{\underset{\sim}{p}}(n)}{\hbar}\,e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)} + e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)}\,\frac{\hat{\underset{\sim}{p}}(n)}{\hbar}\right]. \tag{2.12}$$

Next, we introduce the oscillator boson creation and annihilation operators

$$\hat{b}_j(n) = \frac{1}{\sqrt{2}}\left[\frac{\hat{x}_j(n)}{b_0} + i b_0\,\frac{\hat{p}_j(n)}{\hbar}\right] \tag{2.13a}$$

$$\hat{b}^{+}_j(n) = (\hat{b}_j(n))^{+} \;, \tag{2.13b}$$

where b_0 is the oscillator size parameter and, $j = 1, 2, 3$, label cartesian axes to write:

$$\hat{\underset{\sim}{j}}^c(\underset{\sim}{q}) = \frac{\hbar}{mcb_0}\,\frac{(-i)}{\sqrt{2}}\sum_n \frac{(1+\hat{\tau}_3(n))}{2}\left[e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)}\,\hat{\underset{\sim}{b}}(n) - \hat{\underset{\sim}{b}}^{+}(n)\,e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)}\right], \tag{2.14}$$

where $\hat{\underset{\sim}{b}}^{+}(n)$ is a vector whose components are $(b_1^+(n), b_2^+(n), b_3^+(n))$, with an analogous definition for $\hat{\underset{\sim}{b}}(n)$.

To proceed, we introduce the spherical components of the vector boson creation and annihilation operators $(\hat{b}_{1\mu}^{+}(n), \hat{b}_{1\mu}(n), \mu = 1,2,3)$ which are shown below:

$$\hat{b}_{11}^{+}(n) = -\frac{1}{\sqrt{2}}\left[\hat{b}_1^+(n) + i\,\hat{b}_2^+(n)\right] \tag{2.15a}$$

$$\hat{b}_{1-1}^{+}(n) = \frac{1}{\sqrt{2}}\left[\hat{b}_1^+(n) - i\,\hat{b}_2^+(n)\right] \tag{2.15b}$$

$$\hat{b}_{10}^{+}(n) = \hat{b}_3^+(n) \tag{2.15c}$$

$$\hat{b}_{1\mu}(n) = (-1)^{\mu}\left[\hat{b}_{1-\mu}^{+}(n)\right]^{+} , \tag{2.15d}$$

and we use eq.(2.6) to write:

$$\hat{\underset{\sim}{j}}^{c}(q) = \frac{\hbar}{mcb_0}\sum_{JLM} 4\pi\,(-i)^{L}\,\underset{\sim}{Y}^{*}_{JL;M}(\Omega(q))\,\frac{(-i)}{\sqrt{2}}\sum_{n}\frac{(1+\hat{\tau}_3(n))}{2}\times$$
$$\times\left[\left[\hat{\not{\rho}}_L(n;q)\times\hat{b}_1(n)\right]_{JM} + (-1)^{L+J}\left[\hat{b}_1^+(n)\times\hat{\not{\rho}}_L(n;q)\right]_{JM}\right] , \tag{2.16}$$

where $\underset{\sim}{Y}_{JL;M}(\Omega(q))$ are the vector spherical harmonics,

$$\underset{\sim}{Y}_{JL;M}(\Omega(q)) = \sum_{\mu} C^{L\,1\,J}_{M-\mu\,\mu\,M}\,Y_{LM-\mu}(\Omega(q))\,\xi_{\mu} . \tag{2.17}$$

To find the desired expressions of the convection current multipoles, we have only to expand the vector spherical harmonics in terms of the spherical vector fields, eqs.(2.10).

This can be done in a straightforward way, with the result:

$$\underset{\sim}{Y}_{JL;M}(\Omega(q)) = \sqrt{\frac{2L+1}{2J+1}}\left[C^{L1J}_{000}\,\underset{\sim}{Y}^{(long)}_{JM}(\Omega(q)) + \frac{1}{\sqrt{2}}\left[1-(-)^{L+J}\right]C^{L1J}_{011}\,\underset{\sim}{Y}^{(el)}_{JM}(\Omega(q)) - \right.$$
$$\left. -\frac{1}{\sqrt{2}}\left[1+(-)^{L+J}\right]C^{L1J}_{011}\,\underset{\sim}{Y}^{(mag)}_{JM}(\Omega(q))\right] . \tag{2.18}$$

Now we use eq.(2.18) for the vector spherical harmonics in eq.(2.16), replace the Clebsch–Gordan coefficients by its explicit expressions and compare with eq.(2.8), to find the following expressions for the transverse and longitudinal multipoles of the convection current:

$$\hat{j}^{(long)}_{c;JM}(q) = \frac{\hbar}{mcb_0}\sum_{n}\frac{(1+\hat{\tau}_3(n))}{2}\left[\left[\frac{J}{2(2J+1)}\right]^{1/2}\left[\left[\hat{\not{\rho}}_{J-1}(n;q)\times\hat{b}_1(n)\right]_{JM} - \right.\right.$$
$$\left. -\left[\hat{b}_1^+(n)\times\hat{\not{\rho}}_{J-1}(n;q)\right]_{JM}\right] + \left[\frac{J+1}{2(2J+1)}\right]^{1/2}\left[\left[\hat{\not{\rho}}_{J+1}(n;q)\times\hat{b}_1(n)\right]_{JM} - \right.$$
$$\left.\left. -\left[\hat{b}_1^+(n)\times\hat{\not{\rho}}_{J+1}(n;q)\right]_{JM}\right]\right] , \tag{2.19a}$$

$$\hat{j}^{(el)}_{c;JM}(q) = \frac{\hbar}{mcb_0}\sum_{n}\frac{(1+\hat{\tau}_3(n))}{2}\left[\left[\frac{J+1}{2(2J+1)}\right]^{1/2}\left[\left[\hat{\not{\rho}}_{J-1}(n;q)\times\hat{b}_1(n)\right]_{JM} - \right.\right.$$
$$\left. -\left[\hat{b}_1^+(n)\times\hat{\not{\rho}}_{J-1}(n;q)\right]_{JM}\right] - \left[\frac{J}{2(2J+1)}\right]^{1/2}\left[\left[\hat{\not{\rho}}_{J+1}(n;q)\times\hat{b}_1(n)\right]_{JM} - \right.$$

$$-\left[\hat{b}_1^\dagger(n)\times\hat{\not\rho}_{J+1}(n;q)\right]_{JM}\Bigg]\Bigg] \quad , \tag{2.19b}$$

$$\hat{j}^{(mag)}_{c;JM}(q) = \frac{\hbar}{mcb_0}\frac{(-i)}{\sqrt{2}}\sum_n \frac{(1+\hat{\tau}_3(n))}{2}\left[\left[\hat{\not\rho}_J(n;q)\times\hat{b}_1(n)\right]_{JM}+\left[\hat{b}_1^\dagger(n)\times\hat{\not\rho}_J(n;q)\right]_{JM}\right] . \tag{2.19c}$$

II.4. Longitudinal, transverse electric and transverse magnetic multipoles of the magnetization current

The magnetization current is given by

$$\hat{\underset{\sim}{j}}^{(m)}(\underset{\sim}{x}) = \underset{\sim}{\nabla}\times\underset{\sim}{\mu}(\underset{\sim}{x}) \quad , \tag{2.20}$$

where $\underset{\sim}{\mu}(\underset{\sim}{x})$ is the magnetization density

$$\underset{\sim}{\mu}(\underset{\sim}{x}) = \frac{\hbar}{2mc}\sum_n \hat{g}(n)\,\hat{\underset{\sim}{\sigma}}(n)\,\delta(\underset{\sim}{x}-\hat{\underset{\sim}{r}}(n)) \quad . \tag{2.21}$$

In (2.21), $\hat{g}(n)$ is given by,

$$\hat{g}(n) = g_n\frac{(1-\hat{\tau}_3(n))}{2}+g_p\frac{(1+\hat{\tau}_3(n))}{2} \quad ,$$

where g_n and g_p are, respectively, the neutron and proton magnetic moments in units of the nuclear magneton.

To find the desired expressions of the magnetization current multipoles we consider its Fourier transform,

$$\hat{\underset{\sim}{j}}^{(m)}(\underset{\sim}{q}) = \frac{\hbar}{2mc}\,i\sum_n \hat{g}(n)(\underset{\sim}{q}\times\hat{\underset{\sim}{\sigma}}(n))\,e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)} \quad . \tag{2.22}$$

Remembering that

$$(\underset{\sim}{q}\times\hat{\underset{\sim}{\sigma}}(n)) = -i\sqrt{\frac{8\pi}{3}}\,q\sum_\nu \underset{\sim}{Y}^*_{11;\nu}(\Omega(\underset{\sim}{q}))\,\hat{\sigma}_{1\nu}(n) \quad , \tag{2.23}$$

and using the multipole expansion of $e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)}$, eq.(2.6), the Fourier transform of the magnetization current can be written as:

$$\hat{\underset{\sim}{j}}^{(m)}(\underset{\sim}{q}) = \frac{\hbar}{mcb_0}\sum_{JLM}4\pi(-i)^L\sqrt{\frac{2\pi}{3}}\left[Y_{LM}(\Omega(\underset{\sim}{q}))\times\underset{\sim}{Y}_{11}(\Omega(\underset{\sim}{q}))\right]^*_{JM}\times$$
$$\times\, qb_0\sum_n \hat{g}(n)\left[\hat{\not\rho}_L(n;q)\times\hat{\sigma}_1(n)\right]_{JM} . \tag{2.24}$$

Now we use the expansion

$$\left[Y_{LM}(\Omega(\underset{\sim}{q}))\times\underset{\sim}{Y}_{11}(\Omega(\underset{\sim}{q}))\right]_{JM} = \left[\frac{3(2L+1)}{4\pi(2J+1)}\right]^{1/2}\left[\frac{1}{2}\left[1-(-)^{L+J}\right]C^{L1J}_{011}\,\underset{\sim}{Y}^{(mag)}_{JM}(\Omega(\underset{\sim}{q}))-\right.$$
$$\left.-\frac{1}{2}\left[1+(-)^{L+J}\right]C^{L1J}_{011}\,\underset{\sim}{Y}^{(el)}_{JM}(\Omega(\underset{\sim}{q}))\right] \tag{2.25}$$

to find the following expressions for the magnetization current multipoles:

$$\hat{j}^{(long)}_{m;JM}(q) = 0 \tag{2.26a}$$

$$\hat{j}^{(el)}_{m;JM}(q) = \frac{\hbar}{mcb_0} q b_0 \sum_n \frac{\hat{g}(n)}{2} \left[\hat{\rho}\!\!/_J(n;q) \times \hat{\sigma}_1(n)\right]_{JM} \tag{2.26b}$$

$$\hat{j}^{(mag)}_{m;JM}(q) = \frac{\hbar}{mcb_0} q b_0 \sum_n \frac{\hat{g}(n)}{2} \left[i \left[\frac{J+1}{2J+1}\right]^{1/2} \left[\hat{\rho}\!\!/_{J-1}(n;q) \times \hat{\sigma}_1(n)\right]_{JM} - i \left[\frac{J}{2J+1}\right]^{1/2} \left[\hat{\rho}\!\!/_{J+1}(n;q) \times \hat{\sigma}_1(n)\right]_{JM} \right] . \tag{2.26c}$$

Our aim in this section was to derive eqs.(2.19) and (2.26) for the current multipoles. Looking at these equations we see that, once we find the boson expansion for the charge density multipoles, the boson expansion for the current density multipoles follow immediately.

III. Boson expansion of the charge density multipoles

To find the boson expansion of the charge density multipoles we have first to write $e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)}$ in normal order form. This can be done by expressing it in terms of the oscillator creation and annihilation operators, eq.(2.13), and using the Baker–Campbell–Hausdorff formula to write[2)]

$$e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)} = e^{-b_0^2 q^2/4}\, e^{-\frac{i b_0}{\sqrt{2}}\underset{\sim}{q}\cdot\hat{\underset{\sim}{b}}^{+}(n)}\, e^{-\frac{i b_0}{\sqrt{2}}\underset{\sim}{q}\cdot\hat{\underset{\sim}{b}}(n)} . \tag{3.1}$$

Expanding the exponentials in a power series, we have

$$e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)} = e^{-b_0^2 q^2/4} \sum_{K,K'} \left[-\frac{i b_0}{\sqrt{2}}\right]^{K+K'} \frac{1}{K!K'!} (\underset{\sim}{q}\cdot\hat{\underset{\sim}{b}}^{+}(n))^{K} (\underset{\sim}{q}\cdot\hat{\underset{\sim}{b}}(n))^{K'} . \tag{3.2}$$

Now that we have written $e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)}$ in normal order form, eq.(3.2), our next step is to write it in terms of spherical tensor operators. This is easily accomplished once we write eq.(3.2) in terms of the spherical components of the momentum transfer, $q_{1\mu}$, and of the vector boson creation and annihilation operators, eq.(2.15).

To see how this can be done, consider the generic term $(\underset{\sim}{q}\cdot\hat{\underset{\sim}{b}}^{+}(n))^{K}$. The unitarity of the Clebsch–Gordan coefficients allow us to write it as

$$(\underset{\sim}{q}\cdot\hat{\underset{\sim}{b}}^{+}(n))^{K} = \sum_{\substack{J_2,J_3,\ldots J_{K-1} \\ J_K M_K}} q^{(K)*}_{1J_2\ldots J_{K-1};J_K M_K}\, \hat{b}^{+(K)}_{1J_2\ldots J_{K-1};J_K M_K}(n) , \quad K > 0 . \tag{3.3}$$

where

$$q^{(K)}_{1J_2\ldots J_{K-1};J_K M_K} = \left[q_1 \times \left[\ldots\left[q_1 \times (q_1 \times q_1)_{J_2}\right]_{J_3}\ldots\right]_{J_{K-1}}\right]_{J_K M_K}$$

and

$$\hat{b}^{+(K)}_{1J_2\ldots J_{K-1};J_K M_K}(n) = \left[b_1^{+}(n) \times \left[\ldots\left[b_1^{+}(n) \times \left[b_1^{+}(n) \times b_1^{+}(n)\right]_{J_2}\right]_{J_3}\ldots\right]_{J_{K-1}}\right]_{J_K M_K}$$

with

$$q^{(1)}_{1M} = q_{1M}$$

$$\hat{b}^{+(1)}_{1M}(n) = \hat{b}^{+}_{1M}(n) .$$

Using

$$q_{1M} = \sqrt{\frac{4\pi}{3}}\, q\, Y_{1M}(\Omega(q))$$

and

$$\left[Y_{L_1}(\Omega(\underset{\sim}{q})) \times Y_{L_2}(\Omega(\underset{\sim}{q}))\right]_{LM} = \sqrt{\frac{(2L_1+1)(2L_2+1)}{4\pi(2L+1)}}\, C_{000}^{L_1L_2L}\, Y_{LM}(\Omega(\underset{\sim}{q}))$$

we easily find

$$q^{(K)}_{1J_2\ldots J_{K-1};J_KM_K} = \sqrt{\frac{4\pi}{(2J_K+1)}}\, \mathcal{C}_{1J_2\ldots J_{K-1}J_K}\, q^K\, Y_{J_KM_K}(\Omega(\underset{\sim}{q})) \qquad (3.4)$$

where

$$\mathcal{C}_{1J_2\ldots J_{K-1}J_K} = C_{000}^{11J_2}\, C_{000}^{1J_2J_3} \ldots C_{000}^{1J_{K-1}J_K}$$

$$\mathcal{C}_{1} = 1 \; .$$

Our notation is such that J_ℓ means the ℓ^{th} angular momentum that occur in $\mathcal{C}_{1J_2\ldots J_{K-1}J_K}$. The possible values of J_ℓ are

$$J_\ell = \ell, \ell-2, \ldots 1 \text{ or } 0 \; .$$

Since the components of the vector boson creation operator commute, analogously to eq.(3.4) we have,

$$\hat{b}^{+(K)}_{1J_2\ldots J_{K-1};J_KM_K}(n) = \sqrt{\frac{4\pi}{(2J_K+1)}}\, \mathcal{C}_{1J_2\ldots J_{K-1}J_K}\, (\hat{\underset{\sim}{b}}^+(n)\cdot\hat{\underset{\sim}{b}}^+(n))^{K/2}\, \hat{Y}_{J_KM_K}(\Omega(\hat{\underset{\sim}{b}}^+(n))) \; . \qquad (3.5)$$

Eq.(3.5) suggests to define

$$\hat{Y}^{(K)}_{J_KM_K}(\hat{\underset{\sim}{b}}^+(n)) = (\hat{b}^+(n)\cdot\hat{b}^+(n))^{K/2}\, \hat{Y}_{J_KM_K}(\Omega(\hat{\underset{\sim}{b}}^+(n))) \; . \qquad (3.6)$$

These operators, which are a generalization of the solid harmonics, are homogeneous polinomials of order K in the components of $\hat{\underset{\sim}{b}}^+(n)$ and they transform under rotations as a spherical tensor of order J_K. When $J_K = K$, it reduces to the solid harmonic of order J_K. This can be easily seen by rewriting eq.(3.6) as

$$\hat{Y}^{(K)}_{J_KM_K}(\hat{\underset{\sim}{b}}^+(n)) = (\hat{\underset{\sim}{b}}^+(n)\cdot\hat{b}^+(n))^{(K-J_K)/2}\, \hat{\mathcal{Y}}_{J_KM_K}(\hat{\underset{\sim}{b}}^+(n)) \; ,$$

where $\hat{\mathcal{Y}}_{J_KM_K}(\hat{\underset{\sim}{b}}^+(n))$ is the solid harmonic of order J_K.

Using eqs.(3.4)–(3.6) we find:

$$(\underset{\sim}{q}\cdot\hat{\underset{\sim}{b}}^+(n))^K = \sum_{J_KM_K} A^{(K)}_{J_K}\, \frac{4\pi}{2J_K+1}\, q^K\, Y^*_{J_KM_K}(\Omega(\underset{\sim}{q}))\, \hat{Y}^{(K)}_{J_KM_K}(\hat{\underset{\sim}{b}}^+(n)) \qquad (3.7)$$

where

$$A^{(K)}_{J_K} = \sum_{J_2,\ldots,J_{K-1}} \left[C_{000}^{11J_2}\right]^2 \left[C_{000}^{1J_2J_3}\right]^2 \ldots \left[C_{000}^{1J_{K-1}J_K}\right]^2 \qquad (3.8)$$

with

$$A^{(1)}_1 = 1$$

$$\sum_{J_K} A^{(K)}_{J_K} = 1 \; .$$

Adopting the convention

$$A^{(0)}_0 = 1$$

$$\hat{Y}^{(0)}_{00}(\underset{\sim}{b}^{+}(n)) = \hat{Y}_{00}(\underset{\sim}{b}^{+}(n)) ,$$

the eq.(3.7) is also valid for $K = 0$.

Analogously we find

$$(\underset{\sim}{q}\cdot\hat{\underset{\sim}{b}}(n))^{K} = \sum_{J_K M_K} A^{(K)}_{J_K} \frac{4\pi}{2J_K+1} q^{K} Y^{*}_{J_K M_K}(\Omega(q)) \hat{Y}^{(K)}_{J_K M_K}(\hat{\underset{\sim}{b}}(n)) . \tag{3.9}$$

Thus we can use eqs.(3.2), (3.7) and (3.9) to write:

$$e^{-i\underset{\sim}{q}\cdot\hat{\underset{\sim}{r}}(n)} = \sum_{JM} e^{-b_0^2q^2/4} \sum_{K,K'} \left[-\frac{i b_0 q}{\sqrt{2}}\right]^{K+K'} \frac{1}{K!K'!} \sum_{J_K J'_{K'}}$$

$$\left[\frac{(4\pi)^3}{(2J+1)(2J_K+1)(2J'_{K'}+1)}\right]^{1/2} C^{J_K J'_{K'} J}_{000} A^{(K)}_{J_K} A^{(K')}_{J'_{K'}}$$

$$\left[\hat{Y}^{(K)}_{J_K}(\hat{\underset{\sim}{b}}^{+}(n)) \times \hat{Y}^{(K')}_{J'_{K'}}(\hat{\underset{\sim}{b}}(n))\right]_{JM} Y^{*}_{JM}(\Omega(q)) . \tag{3.10}$$

Given eq.(3.10), the use of eqs.(2.5) and (2.6) leads immediately to:

$$\hat{\rho}_{LM}(q) = e^{-b_0^2q^2/4} \sum_{K,K'} \left[\frac{b_0 q}{\sqrt{2}}\right]^{K+K'} \frac{1}{K!K'!} i^{(L-K-K')} \sum_{J_K,J'_{K'}}$$

$$\left[\frac{(4\pi)}{(2L+1)(2J_K+1)(2J'_{K'}+1)}\right]^{1/2} A^{(K)}_{J_K} A^{(K')}_{J'_{K'}} C^{J_K J'_{K'} L}_{000} \sum_{n}$$

$$\frac{(1 + \hat{\tau}_3(n))}{2} \left[\hat{Y}^{(K)}_{J_K}(\hat{\underset{\sim}{b}}^{+}(n)) \times \hat{Y}^{(K')}_{J'_{K'}}(\hat{\underset{\sim}{b}}(n)) \right]_{LM} \tag{3.11}$$

Therefore the charge density multipole can be written as

$$\hat{\rho}_{LM}(q) = e^{-b_0^2q^2/4} \sum_{K=0}^{\infty} (b_0 q)^{2K+L} \hat{\rho}_{LM}{}^{(2K+L)} \tag{3.12}$$

where

$$\hat{\rho}^{(K)}_{LM} = \sum_{n} \frac{(1 + \hat{\tau}_3(n))}{2} \sum_{K'=0}^{K} \sum_{J'_{K'} J_{K-K'}} Z^{S0(3)}(K'J'_{K'},K-K'J_{K-K'};CL) \times$$

$$\times \left[\hat{Y}^{(K')}_{J'_{K'}}(\hat{\underset{\sim}{b}}^{+}(n)) \times \hat{Y}^{(K-K')}_{J_{K-K'}}(\hat{\underset{\sim}{b}}(n)) \right]_{LM} \tag{3.13}$$

with

$$Z^{S0(3)}(K'J'_{K'},K-K'J_{K-K'};CL) = \frac{(-1)^{(K-L)/2}}{2^{K/2}K'!(K-K')!} \sqrt{\frac{4\pi}{(2L+1)(2J'_{K'}+1)(2J_{K-K'}+1)}} \times$$

$$\times C^{J'_{K'} J_{K-K'} L}_{000} A^{(K')}_{J'_{K'}} A^{(K-K')}_{J_{K-K'}} . \tag{3.14}$$

The eq.(3.14) shows that $Z^{S0(3)}(K'J'_{K'},K-K'J_{K-K'};CL)$ is symmetrical in the interchanges $K' \rightleftarrows K-K'$, $J'_{K'} \rightleftarrows J_{K-K'}$ and is different from zero only if $J'_{K'} + J_{K-K'} + L$ is even, which implies that $K-L$ is even.

In writing eq.(3.12) we used two properties of the charge density multipoles which are, of course, satisfied by eq.(3.11). One is that the even (odd) multipoles have only even (odd) powers of q. The other is that, for a given multipole of order L, the series expansion starts at q^{L}.

In the particular case of the harmonic oscilator shell–model space, where the valence nucleons are in the N^{th} harmonic oscillator shell, the series terminates since, in this case, we can annihilate at most N oscillator quanta and only the boson number conserving terms contribute. As a consequence we have only even multipoles and K and L in eq.(3.12) is restricted to $(2K+L) \leq 2N$.

In this particular case, the expression of $\hat{\rho}_{LM}^{(2K+L)}$ reduces to:

$$\hat{\rho}_{LM}^{(2K+L)} = \sum_n \frac{(1 + \hat{\tau}_3(n))}{2} \sum_{J'_{K+L/2}\, J_{K+L/2}} Z^{SO(3)}\left[K+L/2\, J'_{K+L/2}, K+L/2\, J_{K+L/2}\, ; CL\right] \times \times \left[\hat{Y}^{(K+L/2)}_{J'_{K+L/2}}(\hat{b}^{+}(n)) \times \hat{Y}^{(K+L/2)}_{J_{K+L/2}}(\hat{b}(n))\right]_{LM} . \quad (3.15)$$

Our discussion up to this point have shown that the charge density multipoles can be expanded as $e^{-b_0^2q^2/4}$ times a polynomial in b_0q, where the coefficients of the K^{th} power in the polynomial expansion are the operators $\hat{\rho}_{LM}^{(K)}$. In turn, these operators are given by a normal order expansion in terms of the harmonic oscillator creation and annihilation operators. This expansion has the property that, in a many–body Hilbert space constructed with a finite set of harmonic oscillator single particle wave functions, only a finite number of $\hat{\rho}_{LM}^{(K)}$ has non–null matrix elements and, as a consequence, the series terminates. A particular example is when we restrict the many–body Hilbert space to the harmonic oscillator shell–model space, as shown above.

Eq.(3.14) completely defines the operator $\hat{\rho}_{LM}^{(K)}$ and it can be used to calculate matrix elements of this operator between many–body wave functions. However in what follows we are going to derive an expression for $\hat{\rho}_{LM}^{(K)}$ in terms of coordinates of the

nucleons which gives further insight into the physical meaning of this operator and can be useful in the calculation of its matrix elements.

The derivation starts by comparing eqs.(2.4) and (3.12), which gives:

$$\sum_{K=0}^{\infty} (b_0q)^{2K+L}\, \hat{\rho}_{LM}^{(2K+L)} = \sum_n \frac{(1 + \hat{\tau}_3(n))}{2}\, e^{b_0^2q^2/4}\, j_L(q\, \hat{r}(n))\, \hat{Y}_{LM}(\Omega(\hat{\underline{r}}(n))) .$$

Now, by considering the expansion[7)]

$$e^{z^2/4}\, j_L(xz) = \sum_{K=0}^{\infty} \frac{1}{2^K(2K+2L+1)!!}\, x^L\, L_K^{L+1/2}(x^2)\, z^{2K+L}$$

one finds:

$$\hat{\rho}_{LM}^{(2K+L)} = \sum_n \frac{(1 + \hat{\tau}_3(n))}{2} \frac{1}{2^K(2K+2L+1)!!} L_K^{L+1/2}\left[\left[\frac{\hat{r}(n)}{b_0}\right]^2\right] \left[\frac{\hat{r}(n)}{b_0}\right]^L \hat{Y}_{LM}(\Omega(\hat{\underline{r}}(n))) \quad (3.16)$$

where $L_K^{L+1/2}(x)$ is an associated Laguerre polynomial[7)].

IV. Boson expansion of the current density multipoles

IV.1. Convection current density

To find the boson expansion of the convection current multipoles we use eqs.(2.5) and (3.12) into the eqs.(2.19) to write:

$$\hat{j}^{(long)}_{c;JM}(q) = \frac{\hbar}{mcb_0}\, e^{-b_0^2q^2/4} \sum_{K=0}^{\infty} (qb_0)^{J-1+2K}\, \hat{j}^{(long)(J-1+2K)}_{c;JM} \quad (4.1a)$$

$$\hat{j}^{(el)}_{c;JM}(q) = \frac{\hbar}{mcb_0} e^{-b_0^2 q^2/4} \sum_{K=0}^{\infty} (qb_0)^{J-1+2K}\, \hat{j}^{(el)\,(J-1+2K)}_{c;JM} \tag{4.1b}$$

$$\hat{j}^{(mag)}_{c;JM}(q) = \frac{\hbar}{mcb_0} e^{-b_0^2 q^2/4} \sum_{K=0}^{\infty} (qb_0)^{J+2K}\, \hat{j}^{(mag)(J+2K)}_{c;JM} \tag{4.1c}$$

where

$$\hat{j}^{(long)(J-1+2K)}_{c;JM} = \sum_n \frac{(1+\hat{\tau}_3(n))}{2} \left[\left[\frac{J}{2(2J+1)}\right]^{1/2} \left[\hat{\not{\rho}}^{(J-1+2K)}_{J-1}(n) \times \hat{b}_1(n) - \hat{b}_1^{\dagger}(n) \times \hat{\not{\rho}}^{(J-1+2K)}_{J-1}(n)\right]_{JM} + \left[\frac{J+1}{2(2J+1)}\right]^{1/2} \left[\hat{\not{\rho}}^{(J-1+2K)}_{J+1}(n) \times \hat{b}_1(n) - \hat{b}_1^{\dagger}(n) \times \hat{\not{\rho}}^{(J-1+2K)}_{J+1}(n)\right]_{JM} \right] \tag{4.2a}$$

$$\hat{j}^{(el)\,(J-1+2K)}_{c;JM} = \sum_n \frac{(1+\hat{\tau}_3(n))}{2} \left[\left[\frac{J+1}{2(2J+1)}\right]^{1/2} \left[\hat{\not{\rho}}^{(J-1+2K)}_{J-1}(n) \times \hat{b}_1(n) - \hat{b}_1^{\dagger}(n) \times \hat{\not{\rho}}^{(J-1+2K)}_{J-1}(n)\right]_{JM} - \left[\frac{J}{2(2J+1)}\right]^{1/2} \left[\hat{\not{\rho}}^{(J-1+2K)}_{J+1}(n) \times \hat{b}_1(n) - \hat{b}_1^{\dagger}(n) \times \hat{\not{\rho}}^{(J-1+2K)}_{J+1}(n)\right]_{JM} \right] \tag{4.2b}$$

$$\hat{j}^{(mag)(J+2K)}_{c;JM} = -\frac{i}{\sqrt{2}} \sum_n \frac{(1+\hat{\tau}_3(n))}{2} \left[\hat{\not{\rho}}^{(J+2K)}_{J}(n) \times \hat{b}_1(n) + \hat{b}_1^{\dagger}(n) \times \hat{\not{\rho}}^{(J+2K)}_{J}(n)\right] . \tag{4.2c}$$

When $K = 0$, only the first term survives in eqs.(4.2a) and (4.2b), and the definition of $\hat{\not{\rho}}^{(K)}_{LM}(n)$ follow from eq. 2.5:

$$\hat{\rho}^{(K)}_{LM} = \sum_n \frac{(1+\hat{\tau}_3(n))}{2} \hat{\not{\rho}}^{(K)}_{LM}(n) .$$

Using eq. (3.13) we can easily find the expression of the operators $\hat{\not{\rho}}^{(K)}_{LM}(n)$ and from it and eqs.(4.2) follow the boson expansion of the convection current multipoles. They are:

$$\hat{j}^{(long)(K)}_{c;JM} = \sum_n \frac{(1+\hat{\tau}_3(n))}{2} \sum_{K'=0}^{K+1} \sum_{J'_{K'},\,J_{K+1-K'}} (K+1-2K') \times Z^{SO(3)}(K+1-K'\,J_{K+1-K'}\,K'J'_{K'}\,;CJ) \left[\hat{Y}^{(K')}_{J'_{K'}}(\hat{b}^{+}(n)) \times \hat{Y}^{(K+1-K')}_{J_{K+1-K'}}(\hat{b}(n))\right]_{JM} , \tag{4.3a}$$

$$\hat{j}^{(el)\,(K)}_{c;JM} = \sum_n \frac{(1+\hat{\tau}_3(n))}{2} \sum_{K'=0}^{K+1} \sum_{J'_{K'},\,J_{K+1-K'}} \left[Z^{SO(3)}(K+1-K'\,J_{K+1-K'}\,K'J'_{K'}\,;EJ)_c - Z^{SO(3)}(K'J'_{K'}\,,K+1-K'\,J_{K+1-K'}\,;EJ)_c\right] \times \left[\hat{Y}^{(K')}_{J'_{K'}}(\hat{b}^{+}(n)) \times \hat{Y}^{(K+1-K')}_{J_{K+1-K'}}(\hat{b}(n))\right]_{JM} , \tag{4.3b}$$

$$\hat{j}^{(mag)(K)}_{c;JM} = \sum_n \frac{(1+\hat{\tau}_3(n))}{2} \sum_{K'=0}^{K+1} \sum_{J'_{K'},\,J_{K+1-K'}} \left[Z^{SO(3)}(K+1-K'\,J_{K+1-K'}\,K'J'_{K'}\,;MJ)_c + Z^{SO(3)}(K'J'_{K'}\,,K+1-K'\,J_{K+1-K'}\,;MJ)_c\right] \times (-i) \left[\hat{Y}^{(K')}_{J'_{K'}}(\hat{b}^{+}(n)) \times \hat{Y}^{(K+1-K')}_{J_{K+1-K'}}(\hat{b}(n))\right]_{JM} , \tag{4.3c}$$

The coefficients which appear in eqs. (4.3b) and (4.3c) are given by

$$Z^{SO(3)}(K+1\,J_{K+1}\,,K'\,J'_{K'};EJ)_c = \frac{(-)^{(K+K'+1-J)/2}}{K!\,K'!\,2^{\frac{K+K'}{2}}} \not{\rho}^{(K+1)}_{J_{K+1}} A^{(K')}_{J'_{K'}} \times$$

$$\times \sqrt{\frac{4\pi}{(2J+1)(2J_{K+1}+1)(2J'_{K'}+1)}}\ \frac{1}{2}\left[1+(-)^{J_{K+1}+J'_{K'}+J}\right] C^{J_{K+1}\ J'_{K'}\ J}_{1\ \ \ 0\ \ \ 1}$$

$$Z^{S0(3)}(0\,0\,,\,K+1\ J_{K+1}\,;\,EJ)_c \;=\; 0 \tag{4.4a}$$

and

$$Z^{S0(3)}(K+1\ J_{K+1}\,,\,K'J'_{K'};\,MJ)_c \;=\; \frac{(-)^{(K+K'-J)/2}}{K!K'!2^{(K+K')/2}}\ \mathscr{A}^{(K+1)}_{J_{K+1}}\ A^{(K')}_{J'_{K'}} \times$$

$$\times \sqrt{\frac{4\pi}{(2J+1)(2J_{K+1}+1)(2J'_{K'}+1)}}\ \frac{1}{2}\left[1-(-)^{J_{K+1}+J'_{K'}+J}\right] C^{J_{K+1}\ J'_{K'}\ J}_{1\ \ \ 0\ \ \ 1}$$

$$Z^{S0(3)}(0\,0\,,\,K+1\ J_{K+1}\,;\,MJ)_c \;=\; 0 \tag{4.4b}$$

with

$$\mathscr{A}^{(K+1)}_{J_{K+1}} \;=\; \sum_{J_K} A^{(K)}_{J_K}\ C^{1\ J_K\ J_{K+1}}_{0\,0\ \ \ 0}\ C^{1\ J_K\ J_{K+1}}_{1\,0\ \ \ 1}\ .$$

and where the subscript c means that these coefficients refer to the expansion of the convection current.

The transverse electric coefficients, eq. 4.4a, are different from zero only if $J_{K+1} + J'_{K'} + J$ is even, which implies that $K+1+K'-J$ is also even. In the expansion of the transverse electric multipoles of the convection current, eq. 4.3b, only occur the combination antisymmetrical in the interchange $K+1-K'\left(J_{K+1-K'}\right) \rightleftarrows K'\left(J'_{K'}\right)$. On the other hand, the transverse magnetic coefficients, eq. 4.4b, are different from zero only if $J_{K+1}+J'_{K'}+J$ is odd, which leads to $K+K'-J$ even. In the expansion of the transverse magnetic multipoles, eq. 4.3c, only occur the combination symmetrical in the interchange $K+1-K'\left(J_{K+1-K'}\right) \rightleftarrows K'\left(J'_{K'}\right)$. To derive eqs. 4.4 we performed a recoupling of the angular momentum followed by the use of well-known identities betwee 3–j and 6–j

symbols[8].

In eqs. 4.1 well-known properties of the current density multipoles are explicitly shown. These properties say that the even (odd) longitudinal and transverse electric multipoles have only odd (even) powers of q and that, for a given multipole of order J, the power series starts at q^{J-1}, (except the longitudinal monopole for which the series starts at q). On the other hand, the even (odd) transverse magnetic multipoles have only even (odd) powers of q and, for a given multipole of order J, the power series starts at q^J. These properties follow trivially from eqs. 2.19 once we remember that the polynomial expansion of the even (odd) charge density multipoles have only even (odd) powers of q which starts at q^L for the multipole of order L.

In the case of the harmonic oscillator shell–model space, where the valence nucleons are in the N^{th} harmonic oscillator shell, the series terminates since, in this case, we can annihilate at most N oscillator quanta and only the boson number conserving terms contribute. Thus, the eqs. 4.3 show that K must be odd and the boson expansion of the convection current multipoles reduces to

$$\hat{j}^{(long)(2K+1)}_{c;JM} \;=\; 0 \tag{4.5a}$$

$$\hat{j}^{(el)\,(2K+1)}_{c;JM} \;=\; \sum_n \frac{(1\ +\ \hat{\tau}_3(n))}{2} \sum_{J_{K+1}\ J'_{K+1}} \left[Z^{S0(3)}(K+1\ J'_{K+1}\,,\,K+1\ J_{K+1}\,;\,EJ)_c\ -\right.$$

$$\left.-\ Z^{S0(3)}\,(K+1\ J_{K+1}\ K+1\ J'_{K+1}\ ;\ EJ)_c\right]$$

$$\left[\hat{Y}^{(K+1)}_{J_{K+1}}(\hat{b}^{+}(n)) \times Y^{(K+1)}_{J'_{K+1}}(\hat{b}(n))\right]_{JM} \tag{4.5b}$$

$$\hat{j}^{(mag)(2K+1)}_{c\,;\,JM} \;=\; \sum_n \frac{(1\ +\ \hat{\tau}_3(n))}{2} \sum_{J_{K+1}\ J'_{K+1}} \left[Z^{S0(3)}(K+1\ J'_{K+1}\,,\,K+1\ J_{K+1}\,;\,MJ)_c\ +\right.$$

$$+ Z^{SO(3)}\,(K+1\, J_{K+1}\, K+1\, J'_{K+1}\,;\, MJ)_c\Big]$$

$$(-i)\left[\hat{Y}^{(K+1)}_{J_{K+1}}(\hat{b}^{+}(n)) \times Y^{(K+1)}_{J'_{K+1}}(\hat{b}(n))\right]_{JM} \qquad (4.5c)$$

where $\hat{j}^{(long)(2K+1)}_{C;JM}$ vanishes since, as we can see from eq. 4.3a, the boson number conserving terms do not contribute to the longitudinal multipole of the convection current.

As a consequence we only have even electric and odd magnetic multipoles and the K and J in eqs. 4.1 are restricted to $J + 2K \leq 2N$ for the electric multipoles and $(J + 1 + 2K) \leq 2N$ for the magnetic multipoles. For the electric multipoles we have the further restriction that $\hat{j}^{(el)(J-1)}_{c;JM}$ vanishes identically, since only the term with $J_{J/2} = J'_{J/2} = J/2$ contributes in this case. This results show that the harmonic oscillator shell–model convection current is solenoidal and that the Siegert theorem, which imposes that the low q behaviour of the electric multipoles is q^{J-1} 4), is violated in a qualitative way, as pointed out in reference 2.

Analogous to the case of the charge density multipoles we can find expressions for the boson expansion of the convection current multipoles in terms of the coordinate and momentum of the nucleons which can be useful in the calculation of its matrix elements between many–body states. To show this, we first express the boson creation and annihilation operators in terms of the coordinate and momentum operators to write eqs. 4.2 as:

$$\hat{j}^{(long)(J-1+2K)}_{c;JM} = \frac{i b_0}{2\hbar}\sum_n \frac{(1 + \hat{\tau}_3(n))}{2}\left[\left[\frac{J}{2J+1}\right]^{1/2}\left\{\hat{p}_1(n), \hat{\mathscr{P}}^{(J-1+2K)}_{J-1}(n)\right\}_{JM} + \right.$$

$$\left. + \left[\frac{J+1}{2J+1}\right]^{1/2}\left\{\hat{p}_1(n), \hat{\mathscr{P}}^{(J-1+2K)}_{J+1}(n)\right\}_{JM}\right] \qquad (4.6a)$$

$$\hat{j}^{(el)(J-1+2K)}_{c;JM} = \frac{i b_0}{2\hbar}\sum_n \frac{(1 + \hat{\tau}_3(n))}{2}\left[\left[\frac{J+1}{2J+1}\right]^{1/2}\left\{\hat{p}_1(n), \hat{\mathscr{P}}^{(J-1+2K)}_{J-1}(n)\right\}_{JM} - \right.$$

$$\left. - \left[\frac{J}{2J+1}\right]^{1/2}\left\{\hat{p}_1(n), \hat{\mathscr{P}}^{(J-1+2K)}_{J+1}(n)\right\}_{JM}\right] \qquad (4.6b)$$

$$\hat{j}^{(mag)(J+2K)}_{c;JM} = -\frac{b_0}{2\hbar}\sum_n \frac{(1 + \hat{\tau}_3(n))}{2}\left[\hat{p}_1(n), \hat{\mathscr{P}}^{(J+2K)}_{J}(n)\right]_{JM} \qquad (4.6c)$$

In eqs. 4.6a and 4.6b when $K = 0$ only the first term survives, and the anti–commutator $\left\{\hat{p}_1(n), \hat{\mathscr{P}}^{(K)}_{L}(n)\right\}_{JM}$ means,

$$\left\{\hat{p}_1(n), \hat{\mathscr{P}}^{(K)}_{L}(n)\right\}_{JM} = \sum_\mu C^{1\ \ L\ \ J}_{\mu\ M-\mu\ M}\left[\hat{p}_{1\mu}(n)\hat{\mathscr{P}}^{(K)}_{LM-\mu}(n) + (-1)^{1+L-J}\hat{\mathscr{P}}^{(K)}_{LM-\mu}(n)\,\hat{p}_{1\mu}(n)\right]$$

with an analogous definition for the commutator $[\hat{p}_1(n), \hat{\mathscr{P}}^{(K)}_{L}(n)]_{JM}$.

Using the expression for $\hat{\mathscr{P}}^{(K)}_{LM}(n)$ extracted from the eq. 3.16 in eqs. 4.6 we find the expressions of the operators 4.6 in terms of the coordinate and momentum of the nucleons.

IV.2. Magnetization current density multipoles

Using the eqs. 2.26, the boson expansion of the magnetization current multipoles follow trivially from the boson expansion of the charge density multipoles.

Writing the transverse multipoles as

$$\hat{j}^{(el)}_{m;JM}(q) = \frac{\hbar}{mcb_0}\, e^{-b_0^2q^2/4}\sum_{K=0}^{\infty}(qb_0)^{2K+J+1}\,\hat{j}^{(el)(2K+J+1)}_{m;JM} \qquad (4.7a)$$

$$\hat{j}^{(mag)}_{m;JM}(q) = \frac{\hbar}{mcb_0} e^{-b_0^2 q^2/4} \sum_{K=0}^{\infty} (qb_0)^{2K+J}\, \hat{j}^{(mag)(2K+J)}_{m;JM} \qquad (4.7b)$$

it follows that

$$\hat{j}^{(el)(2K+J+1)}_{m;JM} = \sum_n \frac{\hat{g}(n)}{2} \left[\hat{\wp}^{(2K+J)}_J(n) \times \hat{\sigma}_1(n)\right]_{JM} \qquad (4.8a)$$

$$\hat{j}^{(mag)(2K+J)}_{m;JM} = i \sum_n \frac{\hat{g}(n)}{2} \left[\left[\frac{J+1}{2J+1}\right]^{1/2} \left[\hat{\wp}^{(J-1+2K)}_{J-1}(n) \times \hat{\sigma}_1(n)\right]_{JM} - \left[\frac{J}{2J+1}\right]^{1/2} \left[\hat{\wp}^{(J-1+2K)}_{J+1}(n) \times \hat{\sigma}_1(n)\right]_{JM} \right] . \qquad (4.8b)$$

Thus, from the boson expansion of $\hat{\wp}^{(K)}_{LM}(n)$, we can easily find the boson expansion of the magnetization current multipoles.

In the case of the harmonic oscillator shell–model space the magnetization current multipoles are given by eqs. 4.8, only now $\hat{\wp}^{(K)}_{LM}(n)$ is given by eq. 3.15.

Since when we restrict the charge density multipoles to the harmonic oscillator shell model space we have only even charge density multipoles one sees that we have only even electric and odd magnetic multipoles and K and J in eqs. 4.8 are restricted to $J+2K \leq 2N$ for the electric multipoles and $J-1+2K \leq 2N$ for the magnetic multipoles.

Finally, using the expression of $\hat{\wp}^{(K)}_{LM}(n)$ extracted from eq. 3.16, and the eqs. 4.8 we find an expression of the operators $\hat{j}^{(el)(K)}_{m;JM}$ and $\hat{j}^{(mag)(K)}_{m;JM}$ given explicitly in terms of the coordinate and spins of the nucleons.

V. SU(3) tensor analysis of the charge and current density multipole operators

To be able to do a full shell–model calculation for the matrix elements of the multipole operators in an SU(3) basis, a SU(3) tensor analysis of the one–body operators that occur in the boson expansion of the charge and current density multipoles is required.

That these one–body operators are SU(3) tensors follow from the observation that $\hat{b}^+_1(n)$ and $\hat{b}_1(n)$ are, respectively [10] and [01] SU(3) tensors[1,2]. However, we are interested in tensors that transform according to the subgroup chain $SU(3) \supset SO(3) \supset SO(2)$. The basis states of the irreducible representations in this chain are labelled by the Elliot labels $\lambda\mu kLM$, where λ and μ are SU(3) labels, k is a multiplicity label given by[9]

$$k = \min\{\lambda\mu\} ,\ \min\{\lambda\mu\} - 2 ,\ \ldots 1 \text{ or } 0$$

and

$$L = k ,\ k+1 ,\ \ldots ,\ k + \max\{\lambda\mu\}$$

except when $k = 0$, in which case

$$L = \max\{\lambda\mu\} ,\ \max\{\lambda\mu\} - 2 ,\ 1 \text{ or } 0 \ .$$

To find the SU(3) tensor character of the multipole operators, consider again the generic term

$$\left[\sum_\nu q^*_{1\nu}\, \hat{b}^+_{1\nu}(n) \right]^K .$$

Since $\hat{b}^+_{1\nu}(n)$ and $q_{1\nu}$ are 1ν components of [10] SU(3) tensors,

$$b^{+\,[10]}_{1\nu}(n) = b^{+}_{1\nu}(n)$$

$$q_{1\nu}^{\,[10]} = q_{1\nu}$$

it follows that

$$\sum_{\nu} q^{*}_{1\nu}\, \hat{b}^{+}_{1\nu}(n) = \sum_{\nu} q_{1\nu}^{\,[10]*}\, b^{+\,[10]}_{1\nu}(n) \ .$$

From the unitarity of the SU(3) ⊃ S0(3) ⊃ SO(2) Wigner coefficients and the property that symmetrical polynomials of order λ can couple only to the [λ0] SU(3) representation, the generic term can be written as

$$\left[\sum_{\nu} q^{*}_{1\nu}\, b^{+}_{1\nu}(n) \right]^{K} = \sum_{J_K M_K} q^{[K0]*}_{J_K M_K}\, \hat{b}^{+\,[K0]}_{J_K M_K}(n) \tag{5.1}$$

where

$$q^{[K0]}_{J_K M_K} = \underbrace{\left[q^{[10]} \times q^{[10]} \times \dots \times q^{[10]} \right]}_{K\ \text{terms}}{}^{[K0]}_{J_K M_K} \tag{5.2a}$$

$$b^{+\,[K0]}_{J_K M_K}(n) = \underbrace{\left[b^{+[10]}(n) \times b^{+[10]}(n) \times \dots \times b^{+[10]}(n) \right]}_{K\ \text{terms}}{}^{[K0]}_{J_K M_K} \ . \tag{5.2a}$$

Now remembering the decomposition of the SU(3) ⊃ S0(3) ⊃ SO(2) Wigner coefficients[9)]

$$<[\lambda_1 \mu_1]\, k_1\, L_1\, M_1, [\lambda_2 \mu_2]\, k_2\, L_2\, M_2 \parallel \rho_3\, [\lambda_3 \mu_3]\, k_3\, L_3\, M_3> \; =$$

$$= <[\lambda_1 \mu_1]\, k_1\, L_1, [\lambda_2 \mu_2]\, k_2\, L_2 \parallel \rho_3\, [\lambda_3 \mu_3]\, k_3\, L_3>\; C^{L_1\, L_2\, L_3}_{M_1\, M_2\, M_3} \tag{5.3}$$

where the first term of the second member in eq. 5.3 is the SO(3) isoscalar factor, and the results of the previous section, we can easily show that

$$q^{[K0]}_{J_K M_K} = B^{(K)}_{J_K} \sqrt{\frac{4\pi}{2J_K+1}}\, q^{K}\, Y_{J_K M_K}(\Omega(q)) \tag{5.4}$$

where $B^{(K)}_{J_K}$ is given by

$$B^{(K)}_{J_K} = \sum_{J_2, \dots, J_{K-1}} C^{1\,1\,J_2}_{000} <[10]1\ [10]1 \parallel [20]J_2>\, C^{1\,J_2\,J_3}_{000} <[10]1\ [20]J_2 \parallel [30]J_3> \cdots$$

$$\cdots C^{1\,J_{K-1}\,J_K}_{000} <[10]1\ [K-10]J_{K-1} \parallel [K0]J_K> \tag{5.5}$$

$$B^{(1)}_{1} = 1 \ .$$

In eq. 5.5, as in the previous section, the possible values of J_ℓ are ℓ, ℓ–2, ... 1 or 0. Equally well one has

$$\hat{b}^{+\,[K0]}_{J_K M_K}(n) = B^{(K)}_{J_K} \sqrt{\frac{4\pi}{(2J_K+1)}}\, \hat{Y}^{(K)}_{J_K M_K}(\hat{b}^{+}(n)) \ . \tag{5.6}$$

Putting all this togheter one sees that

$$\left[\sum_{\nu} q_{\nu}^{*}\,\hat{b}_{1\nu}^{+}(n)\right]^{K} = \sum_{J_K M_K} B_{J_K}^{(K)^2}\,\frac{4\pi}{(2J_K+1)}\,Y_{J_K M_K}(\Omega(q))\,q^{K}\,(\Omega(q))\,\hat{Y}_{J_K M_K}^{(K)}(\hat{b}^{+}(n)) \;. \tag{5.7}$$

Comparing eqs. 3.7 and 5.7 one derives the non–trivial identity

$$B_{J_K}^{(K)^2} = A_{J_K}^{(K)} \;. \tag{5.8}$$

Noticing the symmetry property[10],

$$<[10]1\,[\lambda 0]L_1 \,\|\, [\lambda{+}1\;0]L_2> \;=\; <[01]1\,[0\lambda]L_1 \,\|\, [0\lambda{+}1]L_2> \;,$$

we can also easily show that

$$\left[\sum_{\nu} q_{\nu}^{*}\,\hat{b}_{1\nu}(n)\right]^{K} = \sum_{J_K M_K} q_{J_K M_K}^{[K0]^*}\,\hat{b}_{J_K M_K}^{[0K]}(n)$$

where

$$\hat{b}_{J_K M_K}^{[0K]}(n) = \underbrace{\left[\hat{b}^{[01]}(n) \times \hat{b}^{[01]}(n) \times \dots \times \hat{b}^{[01]}(n)\right]}_{K\ \text{terms}}{}^{[0K]}_{J_K M_K} \tag{5.9}$$

The SU(3) tensor $\hat{b}_{J_K M_K}^{[0K]}(n)$ can be written as

$$\hat{b}_{J_K M_K}^{[0K]}(n) = B_{J_K}^{(K)}\sqrt{\frac{4\pi}{2J_K+1}}\;\hat{Y}_{J_K M_K}^{(K)}(\hat{b}(n)) \;.$$

These results are all that we need to write the charge and current density multipoles in terms of SU(3) tensors.

5.1. Charge density multipoles

Eq. (3.13) for the $\hat{\rho}_{LM}^{(K)}$ can be written using the results of this section, as:

$$\hat{\rho}_{LM}^{(K)} = \sum_{K'=0}^{K} \frac{i^{L-K}}{2^{K/2}K'!(K-K')!} \sum_{J'_{K'} J_{K-K'}} \sqrt{\frac{1}{4\pi(2L+1)}}\; C_{000}^{J'_{K'} J_{K-K'} L} \times$$
$$\times\, B_{J'_{K'}}^{(K')} B_{J_{K-K'}}^{(K-K')} \sum_{n} \frac{(1+\hat{\tau}_3(n))}{2}\left[\hat{b}_{J'_{K'}}^{+[K'0]}(n) \times \hat{b}_{J_{K-K'}}^{[0\;K-K']}(n)\right]_{LM} \tag{5.10}$$

The SU(3) coupling of an [s0] and [0s'] SU(3) tensors is multiplicity free and given by[11]

$$[s0] \times [0s'] = \sum_{p=0}^{\min\{s,s'\}} [s{-}p, s'{-}p] \;.$$

Therefore one has,

$$\hat{\rho}_{LM}^{(K)} = \sum_{K'=0}^{K} \sum_{p=0}^{\min\{K',K-K'\}} \sum_{k} \Delta([K'0][0\;K{-}K'] \,\|\, [K'{-}p\;K{-}K'{-}p]k;\,CL) \times$$
$$\times \sum_{n} \frac{(1+\hat{\tau}_3(n))}{2}\left[\hat{b}^{+[K'0]}(n) \times \hat{b}^{[0\;K-K']}(n)\right]_{kLM}^{[K'-p\;K-K'-p]} \tag{5.11}$$

where

$$\Delta([K'0][0\,K-K'] \,\|\, [K'-p\;K-K'-p]k;\,CL) =$$

$$= \sum_{L'_{K'}L_{K-K'}} Z^{SU(3)}(K'L'_{K'}, K-K'\;L_{K-K'};\,CL) < [K'0]L'_{K'}, [0\;K-K']L_{K-K'} \,||\, [K'-p\;K-K'-p]kL> \tag{5.12}$$

where the expression of $Z^{SU(3)}\,(K'L'_{K'}\;K-K'\;L_{K-K'};\,CL)$ is identical to the one of $Z^{SO(3)}\,(K'L'_{K'}\;K-K'\;L_{K-K'};\,CL)$ with the replacement $\sqrt{\frac{4\pi}{2L'_{K'}+1}}\,A^{(K')}_{L'_{K'}} \to B^{(K')}_{L'_{K'}}$ and $\sqrt{\frac{4\pi}{2L_{K-K'}+1}}\,A^{(K-K')}_{L_{K-K'}} \to B^{(K-K')}_{L_{K-K'}}$.

In the nuclear harmonic oscillator shell model space only the number conserving terms contribute. In this case K is an even number and $\hat{\rho}^{(K)}_{LM}$ reduces to

$$\hat{\rho}^{(K)}_{LM} = \sum_{p=0}^{K/2} \sum_{k} \Delta\left(\left[\tfrac{K}{2}0\right]\left[0\,\tfrac{K}{2}\right] \,\Big\|\, \left[\tfrac{K}{2}-p\;\tfrac{K}{2}-p\right]k;\,CL\right) \times$$
$$\times \sum_{n} \frac{(1 + \hat{\tau}_3(n))}{2} \left[\hat{b}^{+}(n)^{[\frac{K}{2}0]} \times \hat{b}(n)^{[0\,\frac{K}{2}]}\right]^{[\frac{K}{2}-p\;\frac{K}{2}-p]}_{kLM} \tag{5.13}$$

5.2 Current density multipoles

Here we only give a brief discussion of how to perform the SU(3) tensor analysis of the current density multipoles.

Using the results of section 5.1 we can write the operators $\hat{j}^{(el)(K)}_{C;JM}$ as

$$\hat{j}^{(el)(K)}_{C;JM} = \sum_{n} \frac{(1 + \hat{\tau}_3(n))}{2} \sum_{K'=0}^{K+1} \sum_{J'_{K'}J_{K+1-K'}} \cdot \Big[Z^{SU(3)}(K+1-K'\;J_{K+1-K'},\,K'J'_{K'};\,EJ)_c -$$
$$- Z^{SU(3)}(K'J'_{K'},\,K+1-K'\;J_{K+1-K'};\,EJ)_c\Big] \left[\hat{b}^{+[K'0]}_{J'_{K'}}(n) \times \hat{b}^{[0\;K+1-K']}_{J_{K+1-K'}}(n)\right]_{JM} \tag{5.14}$$

where the coefficient $Z^{SU(3)}(K+1\;J_{K+1},\,K'J'_{K'};\,EJ)_c$ is given by an expression identical to $Z^{SO(3)}(K+1\;J_{K+1},\,K'J'_{K'};\,EJ)_c$ with the replacement $\sqrt{\frac{4\pi}{2J_{K+1}+1}}\,\mathscr{A}^{(K+1)}_{J_{K+1}} \to \mathscr{B}^{(K+1)}_{J_{K+1}}$ and $\sqrt{\frac{4\pi}{2J'_{K'}+1}}\,A^{(K')}_{J'_{K'}} \to B^{(K')}_{J'_{K'}}$ where

$$\mathscr{B}^{(K+1)}_{J_{K+1}} = \sum_{J_K} B^{(K)}_{J_K}\, C^{1\;J_K\;J_{K+1}}_{1\;0\;\;1} < [10]1\;[K0]J_K \,||\, [K+1\;0]J_{K+1}> . \tag{5.15}$$

The last step consists of performing the SU(3) coupling to get:

$$\hat{j}^{(el)(K)}_{C;JM} = \sum_{K'=0}^{K+1} \sum_{p=0}^{\min\{K',K+1-K'\}} \sum_{k} \Delta([K'0][0\;K+1-K'] \,\|\, [K'-p\;K+1-K'-p]k;\,EJ)_c \times$$
$$\times \sum_{n} \frac{(1 + \hat{\tau}_3(n))}{2} \left[\hat{b}^{+[K'0]}(n) \times \hat{b}^{[0\;K+1-K']}(n)\right]^{[K'-p\;K+1-K'-p]}_{kJM} \tag{5.16}$$

where

$$\Delta([K'0][0\;K+1-K'] \,\|\, [K'-p\;K+1-K'-p]k;EJ)_c =$$

$$\sum_{J_{K+1-K'}J'_{K'}} \Big[Z^{SU(3)}(K+1-K'J_{K+1-K'},K'J'_{K'};\,EJ)_c - Z^{SU(3)}(K'J'_{K'},\,K+1-K'J_{K+1-K'};EJ)_c\Big] \times$$

$$< [K'\;0]J'_{K'}\;[0\;K+1-K']J_{K+1-K'} \,\|\, [K'-p,\;K+1-K'-p]kJ> .$$

The steps necessary to find a corresponding expression for $j^{(mag)(K)}_{c\,;\,JM}$ are analogous to the ones above and here we only present the final result. It is:

$$j^{(mag)(K)}_{c\,;\,JM} = \sum_{K'=0}^{K+1} \sum_{p=0}^{\min\{K',K+1-K'\}} \sum_{k} \Delta([K'0][0\ K+1-K'] \,\|\, [K'-p\ K+1-K'-p]k;\ MJ)_c \times$$

$$\times\,(-i) \sum_{n} \frac{(1\ +\ \hat{\tau}_3(n))}{2} \left[\hat{b}^{+[K'0]}(n) \times \hat{b}^{[0\ K+1-K']}(n)\right]^{[K'-p\ K+1-K'-p]}_{kJM}$$

where

$$\Delta([K'0][0\ K+1-K']\,\|\,[K'-p\ K+1-K'-p]k;\ MJ)_c =$$

$$\sum_{J_{K'},J_{K+1-K'}} \left[Z^{SU(3)}(K+1-K'J_{K+1-K'},K'J_{K'};MJ)_c + Z^{SU(3)}(K'J_{K'},K+1-K'J_{K+1-K'};MJ)_c\right] \times$$

$$\times < [K'\ 0]J_{K'}\ [0\ K+1-K']J_{K+1-K'}\,|\,[K'-p,\ K+1-K'-p]kJ >$$

where to find the expression of $Z^{SU(3)}(K+1\ J_{K+1}\ ,\ K'\ J_{K'};\ MJ)_c$ we start with the expression of $Z^{SO(3)}(K+1\ J_{K+1}\ ,\ K'\ J_{K'};\ MJ)_c$ and do the same replacement as in $Z^{SU(3)}(K+1\ J_{K+1}\ ,\ K'\ J_{K'};\ EJ)_c$.

In the harmonic oscillator shell model space only the boson number conserving terms contribute. As a consequence K is an odd number and the transverse electric and transverse magnetic multipoles of the convection current reduces to:

$$\hat{j}^{(el)(2K+1)}_{c;JM} = \sum_{p=0}^{K+1} \sum_{n} \Delta\left[[K+1\ 0]\ [0\ K+1]\,\|\,[K+1-p\ K+1-p]k;\ EJ\right]_c \times$$

$$\times \sum_{n} \frac{(1\ +\ \hat{\tau}_3(n))}{2} \left[\hat{b}^{+}(n)^{[K+1\ 0]} \times \hat{b}(n)^{[0\ K+1]}\right]^{[K+1-p\ K+1-p]}_{kJM}$$

$$\hat{j}^{(mag)(2K+1)}_{c\,;\,JM} = \sum_{p=0}^{K+1} \sum_{n} \Delta\left[[K+1\ 0]\ [0\ K+1]\,\|\,[K+1-p\ K+1-p]k;\ MJ\right]_c \times$$

$$\times\,(-i) \sum_{n} \frac{(1\ +\ \hat{\tau}_3(n))}{2} \left[\hat{b}^{+}(n)^{[K+1\ 0]} \times \hat{b}(n)^{[0\ K+1]}\right]^{[K+1-p\ K+1-p]}_{kJM} .$$

Given the tensor analysis of the charge density multipoles, eq. 5.13, the SU(3) tensor analysis of the multipoles of the magnetization current follow in a straightforward way, once we use eqs. 2.26.

VI. Conclusions

In this paper we have derived a formally exact expansion for the charge and current density multipoles. This expansion is given by a gaussian times a power series on the momentum transfer q , where the coefficients of the powers of q are specific one–body operators. These one–body operators are normal–ordered homogeneous polynomials of the harmonic–oscillator boson creation and annihilation operators. Therefore, one separates the dependence on the momentum transfer, given by the prescribed functions of q , from the dependence on the nuclear structure, given by the matrix elements of the one–body operators. In the special case of the harmonic oscillator shell–model space the series terminates and only the boson number conserving terms of the one–body operators survive. As a consequence, there are form factors whose shape is independent of the nuclear structure, as pointed out in ref. 2 for the special case of the sd–shell. For a N–shell nucleus, eqs. 3.15, 4.5 and 4.8 show that the shape of the form factors[4] $F^L_{0^+ \to 2N^+}(q)^2$, $F^T_{0^+ \to 2N^+}(q)^2$ and $F^T_{0^+ \to 2N+1^+}(q)^2$ are independent of the nuclear structure. Besides, the last two form factors depend only on the magnetization current. Also, as shown in ref. 2 for the special case of the sd–shell, the harmonic oscillator shell model current is purely trasverse and, as a consequence, it violates the Siegert theorem in a qualitative way. Thus we see that, if one takes for the nuclear current, the sum of the convection and magnetization currents, the $B(EJ; 0^+ \to J^+)$ depends only on configurations outside the harmonic oscillator shell model space. This, we think, indicates the need to consider expressions for the transverse electric multipoles which incorporates the constraints of the Siegert theorem[12].

Expansions of the form factors[2] of the type derived here are well–known in the literature and widely used in a phenomenological analysis of the data[13]. They have been derived by a direct expansion of the matrix elements of the multipole operators in the harmonic oscillator basis[14]. In this paper we have shown that, alternatively, it can be derived by a normal ordered expansion of the multipole operators in terms of harmonic oscillator bosons. This procedure has many advantages. One is that it is much simple to find the consequences for the form factors, of the restrictions of the many–particle Hilbert space. Other is that it exhibits the general structure of the expansion, valid for any nuclei and any state of a given nucleus. Also, what is very important, it shows that the expansion can have a greater generality then we would suppose. Indeed, the sucess of the phenomenological analysis of the data[13] indicates that the boson expansion has a much greater generality, being valid even when the restriction to the harmonic oscillator shell model space is not appropriate. One example of this fact is when a renormalization of the shell–model form factors in sufficient to describe the data. When this is true one sees that the expansion of the multipole operators has the number of terms required by the h.o. shell

model restriction, however, the value of the coefficients depends (in most cases very strongly) on configurations outside the h.o. shell model space.

Finally we have performed an SU(3) tensor analysis of the charge and current density multipoles which is of considerable pratical importance for calculations with shell–model wave functions in a SU(3) basis.

I would like to thank Dr. D. Rowe for discussions in the beginning of this work and Dr. P. Rochford for calling my attention to reference 11.

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