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MICROSCOPIC THEORY OF SUPERFLUID HELIUM-4

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MICROSCOPIC THEORY OF SUPERFLUID

HELIUM-4

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ABSTRACT

To understand the properties of superfluid Helium-4 we propose a microscopic theory for the description of this imperfect boson gas at very low temperature. Our microscopic theory contains only two parameters that are inferred from the He-He interaction potential. By using this microscopic theory we get a very good description of superfluid Helium-4. Our approach is based on Bose-Einstein condensation and for its description we use field theory at finite temperatures. We have obtained, in a simple way, many physically relevant quantities. The most relevant achievement was the determination of the roton spectrum with Landau's parameters in good agreement with experiment. We get also good results for the critical temperature and a fairly accurate picture of the nature of the two-fluid components.

I. INTRODUCTION

In this paper we shall study a system of boson particles which interact among themselves through binary forces such that the interparticle potential has finite range and exhibits no two particle bound state. A dilute system of particles whose two-body potential satisfy the above properties is defined as an imperfect gas.

The central feature of an imperfect gas is that the description of such a system at very low temperatures is roughly insensitive to the details of the interatomic forces. The system, under these circumstances is sensitive only to some general features of the potential and, consequently, the complete description of an imperfect gas could be achieved by employing just three parameters^(1,2). Two of them are dynamical parameters and the other one is a kinematical parameter.

The imperfect gas problem can be formulated as the problem of achieving a description of the system at extremely low temperature in terms of these (three) parameters. This would ultimately require a construction of an effective Hamiltonian that contains the essential features of the exact Hamiltonian and that allows for a systematic method for the calculation of the partition function of the system. The effective theory should not contain free parameters since if one is able to infer from the interaction potential its relevant features one should also be able to infer from it the relevant physical constants.

What are the details of the interatomic potential which are relevant in order to understand an imperfect boson gas at low temperature? That is, to what features of the potential would the system be sensitive in this limit? We argue that the system is sensitive to a global aspect of the potential (the scattering length) and to a local aspect of the potential (the depth of the potential).

The first dynamical parameter is the scattering length — a — defined by

$$a = \frac{m}{4\pi} \int V(\vec{r}) d^3\vec{r} \quad (1.1)$$

where V is the two-body interaction potential of the atoms.

From expression (1.1) one can see that the scattering length is just the scattering amplitude in Born's approximation in the zero-momentum limit. In this limit the scattering amplitude is not sensitive to the local aspects of the potential but rather to a global aspect of it. The total cross section at low energies assumes the form of a geometrical cross section, $\sigma = 4\pi(a)^2$. In this way a is, roughly speaking, the effective diameter of the potential. The scattering looks like that from a hard sphere of diameter a .

The choice of the depth of the potential as another relevant parameter for the description of the system at very low temperatures can be understood within the single particle context. For a particle under the action of an external force, then, at low temperatures, the particle tends to seat at the most stable configuration. That is, the minimum of the potential. Within this classical reasoning the only relevant region of the potential at low temperatures would be the minimum of the potential. Furthermore, within the semiclassical approximation the energy will be given, in the zero-temperature limit, by the depth of the potential.

The kinematical parameter is the atom mass: m .

In the case of He^4 the scattering length — a — and the depth of the potential — ΔE — can be read from the interatomic potential. The approximate values for these parameters for a Lennard-Jones potential are⁽³⁾

$$\begin{aligned} m &\cong 3,5 \cdot 10^9 \text{ eV} \\ \Delta E &\cong 10,2 \cdot 10^{-4} \text{ eV} \\ a^{-1} &\cong 10^3 \text{ eV} \end{aligned} \quad (1.2)$$

where in (1.2), as well as throughout our paper, we are using the natural system of units ($k = 1, c = 1, \hbar = 1$).

Although these parameters differs from each other by at least 6 orders of magnitude they give, as we will show in this paper, a fairly good account of all relevant physical quantities in Helium superfluid phase. The approach used by us makes use

of Bose-Einstein condensation⁽⁴⁻⁶⁾ in order to derive all relevant features of the superfluidity phenomena.

Our approach allows for the determination of the partition function, and ultimately the whole thermodynamics of superfluid Helium-4 in terms of these parameters. In our approach there are, then, no free parameters. We have been particularly successful in getting the following set of physical quantities

ρ_{He}	— Helium density
v_s	— sound speed
ϵ	— Helium binding energy
p_0	— momentum of the roton spectrum
Δ	— energy gap
μ	— "roton mass"
$P(0)$	— pressure at zero temperature

This paper is organized as follows:

In Chapter II we present some relevant features that a microscopic theory should account for in order to describe superfluid Helium. The features presented here might not be complete but certainly it covers a fairly large amount of relevant experimental data or phenomenological aspects of the problem.

In Chapter III we construct the relevant effective Hamiltonian and review the basic framework for dealing with Bose-Einstein condensation.

The method employed by us makes use of a field theoretical approach fully developed in ref. (5). The relevant expressions for our work, developed in ref. (5), are also presented in this Chapter.

In Chapter IV we compute explicitly, up to the one loop approximation in field theory, the relevant thermodynamical potential for our effective Hamiltonian.

The description of the superfluid at very low temperatures is presented in Chapter V. In this Chapter we give an account of some of the features presented in Chapter II.

The two fluid description is presented in Chapter VI. We have been concerned with the identification of the normal component and in presenting an alternative method for making calculations of thermodynamical variables for the normal and superfluid components.

In Chapter VII we have been mainly concerned with simple estimates of the critical temperature. We have made simple calculations and, by making the gap equal to zero, we have got a good result for T_λ .

We end this paper with Chapter VIII which is devoted to Conclusions.

II. LIQUID HELIUM AT LOW TEMPERATURES⁽⁷⁻⁹⁾

II.1. Excitation Spectrum

Liquid Helium-4 exhibits two distinct phases. In the phase I Helium-4 behaves as a normal liquid whereas in phase II it behaves as a superfluid. Helium-4 undergoes a phase transition from phase I to phase II at a temperature $T_\lambda = 2.18^\circ\text{K}$ (λ -point). In phase II one can distinguish two regimes. For temperatures in the range $1.0^\circ\text{K} < T < 2.18^\circ\text{K}$ the liquid behaves as a mixture of a "normal component" and a "superfluid component" (anomalous region). Below 1°K the normal component is negligible.

It has been stressed by Landau that the properties of the two regions below the λ -point, can be understood if one looks at the dispersion relation satisfied by the excitations in He II. The excitation that is relevant at very low temperatures is the phonon, satisfying the dispersion relation

$$\omega^{\text{ph}}(p) = v_s p \quad (2.1)$$

where v_s is the sound speed and p is the phonon momentum. The superfluidity property can be understood from the dispersion relation (2.1). Superfluidity is then closely related to the existence of phonons.

The other type of quantum excitation, relevant in understanding the behavior of He^4 in the two fluid region, was named by Landau^{10,11}, its discoverer, a roton. The roton dispersion relation was postulated by Landau to be of the form

$$\omega^{\text{roton}}(p) = \frac{(p - p_0)^2}{2\mu} + \Delta \quad (2.2)$$

where Δ (the energy gap), μ (the effective mass) and p_0 are phenomenological constants that could be obtained, in principle, from a microscopic theory.

The spectrum of excitations should be inferred from the solutions of the quantum mechanical many body problem. For an aggregate of roughly 10^{22} particles this is

an almost impossible task. In spite of this Feynman showed⁽¹²⁾ that one can throw considerable light on the problem by using physical intuition to obtain a qualitative picture of the wave functions. In Feynman's approach the dispersion relation of the excitations can be inferred from the form factor for scattering of neutrons from the liquid and thus can be obtained experimentally.

The derivation of the roton spectrum from first principles is thus a long standing problem in theoretical physics.

II.2. Other Features of He II

Although the spectrum of excitations is by far the most relevant problem to be settled for superfluid Helium-4, there are yet other features that any microscopic theory should account for. These features are the following:

II.2.a. Ground state energy

The ground state of liquid Helium is a N-body bound state that has a self-determined equilibrium density in the absence of external pressure. One should be able to write, for the energy of the ground state

$$E = E(\rho) \quad (2.3)$$

and find the equilibrium density from the condition

$$\left. \frac{dE}{d\rho} \right|_{\rho = \rho_0} = 0 \quad (2.4)$$

For liquid He, $E(\rho_0)$ is the binding energy of the particles and is equivalent to the latent heat of vaporization at $T = 0^\circ\text{K}$.

II.2.b. Binding energy⁽⁸⁾

Since Helium-4 is a N-body bound state, liquid Helium has a binding energy per atom at absolute zero. This binding energy should be determined from the condition

$$\varepsilon = \frac{E(\rho_0)}{N} \quad (2.5)$$

with ρ_0 given by (2.4).

The binding energy is an example of Helium phenomena for which the two-fluid model alone is simply incapable of providing a suitable description. In his book⁽⁸⁾, London discussed various approaches to this problem.

The values for the Helium binding energy is⁽⁸⁾

$$\varepsilon = -13 \frac{\text{cal}}{\text{mol}} = -5.5 \frac{10^{-4} \text{ eV}}{\text{Atom}} \quad (2.6)$$

II.2.c. Helium density

In some approaches^(1, 2) to the imperfect Bose gas system at very low temperatures the Helium density, given by

$$\rho_{\text{He}} = 1.6 \cdot 10^8 (\text{eV})^3 \quad (2.7)$$

or the interatomic distance d

$$d^{-1} = (\rho_{\text{He}})^{1/3} \quad (2.8)$$

is taken to be a fundamental parameter.

As explained below, in order to understand, from a microscopic theory, why Helium is liquid at very low temperatures it is also essential to get from this theory the Helium density. In principle this density should also be inferred from (2.4), that is

$$\rho_{\text{He}} = \rho_0 \quad (2.9)$$

II.2.d. Quantum liquid

Why is He^4 still liquid at $T = 0$ at zero external pressure?

Tisza's approach⁽¹³⁾ to this problem imposes two postulates one of which is analogous to (2.4).

Since the He-He interaction potential is shallow and has short range then, as a result of the small Helium mass, zero-point fluctuations prevents Helium-4 of being a solid (the lattice "melts"). If on the other hand we get from a microscopic theory the Helium density given by (2.9), then from (2.8) we should conclude that it cannot be a gas (the interparticle distance is typical of a solid).

In this way, by looking at the features of the potential and from (2.9) we should understand why Helium is liquid at zero temperature.

II.2.e. Critical velocity

Superflow in Helium liquid is subject to a restriction which limits the velocity of the superfluid to values smaller than a "critical velocity". An upper bound to this critical velocity can be obtained from the knowledge of the excitation spectrum. This upper bound is⁽¹⁰⁾

$$v_c < \left(\frac{\omega(\vec{p})}{p} \right)_{\min} \quad (2.10)$$

where $(\omega(\vec{p})/p)_{\min}$ stands for the condition

$$\left. \frac{d}{dp} \left(\frac{\omega(\vec{p})}{p} \right) \right|_{p=p_{\min}} = 0 \quad (2.11)$$

The determination of the spectrum is thus important in order to get expressions for the superfluid flow velocity limit.

II.2.f. Two-fluid description

As pointed out earlier, below the λ -point Helium-4 behaves as a mixture of two fluids. Below 1.1°K the so-called normal component is negligible.

There are many phenomenologically successful applications of the two-fluid model. However the understanding of the two-fluid nature as well as a well defined scheme for computing the thermodynamical properties of normal and/or superfluid components has not been achieved yet.

In this context it is important to recall that the proposers of the two-fluid model were not in agreement with regard to the nature of the two-fluid⁽¹⁴⁾. Whereas in Landau's view the normal component is associated to the excitations in Tisza's view the normal component was composed of Helium atoms.

II.2.g. Determination of the λ -point

Any attempt to describe Helium-4 should predict a transition at $T_\lambda = 2.18^\circ\text{K}$. The determination of the critical temperature is a longstanding problem in superfluid Helium-4.

In the pioneering work of London, in which he suggested that the λ -transition can be attributed to the macroscopic occupation of the $\vec{p} = 0$ state (Bose-Einstein condensation), he made the first successful prediction for T_λ . London's result was

$$T_c^L = 3.13^\circ\text{K} \quad (2.12)$$

II.2.h. Abnormal transport coefficients

It is known that the heat conductivity of superfluid Helium just below the λ -point is 5 orders of magnitude larger than the normal liquid (He^I). This large heat conductivity prevents the establishment of temperature differences in superfluid Helium and this is the basis for understanding some surprising properties of Helium-4⁽⁹⁾.

The viscosity coefficient of the superfluid component is negligible. The viscosity coefficient of the fluid as a whole suffers a drop of about 1/3 of the normal liquid just below the λ -point.

II.2.i. Thermomechanical and mechanocaloric effects

Thermomechanical and mechanocaloric effects are among the most remarkable properties of superfluid Helium. The fact that a temperature difference (ΔT) gives rise to a pressure difference (ΔP) is the basic feature of the thermomechanical effect. The relation between ΔP and ΔT has been deduced by London in 1939 on the

basis of essentially reversible thermodynamics. His result is

$$dP = \frac{S}{V} dT \quad (2.13)$$

III. EFFECTIVE HAMILTONIAN AND THE APPROACH TO BOSE-EINSTEIN CONDENSATION

III.1. Effective Hamiltonian

In order to get the thermodynamics of an imperfect gas we shall replace the Hamiltonian of the system by an effective Hamiltonian. Within the field theoretical method this amounts to making the replacement

$$H_{\text{int}}[\psi] \rightarrow H_{\text{eff}}[\psi] \quad (3.1)$$

where ψ is the complex boson field and $H_{\text{int}}[\psi]$ can be written, for particles that interact through binary forces whose potential is $V(\vec{x} - \vec{x}')$, as⁽¹⁸⁾

$$\begin{aligned} H_{\text{int}}[\psi] &= \frac{1}{2} \int d^3\vec{x} \int d^3\vec{x}' \psi^*(\vec{x}, \tau) \psi(\vec{x}, \tau) V(\vec{x} - \vec{x}') \psi^*(\vec{x}', \tau) \psi(\vec{x}', \tau) \\ &= \frac{1}{2} \int d^3\vec{x} \int d^3\vec{x}' \rho(\vec{x}, \tau) V(\vec{x} - \vec{x}') \rho(\vec{x}', \tau) \end{aligned} \quad (3.2)$$

where

$$\rho \equiv \psi^*(\vec{x}, \tau) \psi(\vec{x}, \tau) \quad (3.3)$$

The justification for the replacement (3.1) is that the scattering of particles at low momenta is insensitive to the shape of the potential^(1,2). Within the shape independent approximation we can replace the potential by any other having the same scattering length.

In order to find the effective Hamiltonian suited to He^4 at very low temperatures we show in fig. (1) the two particle interaction potential between two Helium atoms. The essential feature of the two-body potential is that, as far as the repulsive part of the potential is concerned, the potential is typical of a hard sphere whose diameter is of the order^(2,15) 2\AA . We take then

$$a \simeq 2\text{\AA} = (10^3 \text{ eV})^{-1} \quad (3.4)$$

One is then tempted to make the replacement (3.1) with H_{eff} given by

$$H_{\text{eff}}[\psi] = \frac{\lambda}{2} \int d^3\vec{x} (\psi^*(\vec{x}, t) \psi(\vec{x}, t))^2. \quad (3.5)$$

The parameter λ is then fixed by condition (1.1) yielding

$$\lambda = \frac{8\pi a}{m}. \quad (3.6)$$

The hard-sphere boson gas described by (3.5) have been extensively investigated in a series of papers by Lee, Yang and Huang^(1,16). If the scattering length is fixed by (3.6) then apparently there is no more room for adding terms to the effective Hamiltonian. However the scattering length fixed criterion together with the requirement of particle number conservation allows for yet another term. The most general effective Hamiltonian preserving $U(1)$ symmetry and the scattering length as given by (3.6) is

$$\mathcal{H}_{\text{eff}} = -\Delta E \psi^* \psi + \frac{4\pi a}{m} (\psi^* \psi)^2. \quad (3.7)$$

where ΔE in (3.10) is a term with energy dimension. In appendix A we present a naive argument that allows us to identify ΔE with the depth of the potential

$$\Delta E = M_{\text{in}}(V(r)) \cong 10.2 \cdot 10^{-4} \text{ eV}. \quad (3.8)$$

Model (3.7) is an attempt to take into account the attractive nature of the potential which has been always overlooked. The attractive nature of the potential has been replaced by an "optical approximation". Within this optical approximation one has a description of hard spheres such that, between collisions, the "free" particles ($\lambda = 0$ in (3.7)) moving through the system sees an uniform potential of an appropriate depth. The idea is that, between collisions, the effect of the rest of the system can be replaced by a medium having an index of refraction defined by the minimum of the potential.

Model (3.7) is a prototype of theories that exhibits spontaneous breakdown of symmetry. The symmetry that is spontaneously broken in model (3.7) is the $U(1)$ symmetry that, as is well known⁽¹⁷⁾, is the symmetry that is broken in the superfluid phase of He-4. One can then predict two phases for He⁴. The one for which the $U(1)$ symmetry is broken corresponds to superfluid He⁴ at very low temperatures.

III.2. Field Theoretical Approach^(5,18,19)

Within the field theoretical approach one writes the partition function as a sum over field theoretical configurations satisfying periodic boundary conditions⁽¹⁹⁾

$$\psi(\vec{x}, \tau) = \psi(\vec{x}, \tau + \beta) \quad (3.9)$$

as

$$Z = \iint \mathcal{D}\psi \mathcal{D}\psi^* e^{-\int_0^\beta d\tau \int d^3\vec{x} \mathcal{L}[\psi, \partial_\mu \psi]} \quad (3.10)$$

where β in (3.9) and (3.10) is the inverse of the temperature $\beta = 1/T$.

The Lagrangian density \mathcal{L} can be written as a kinetic term (K) plus an interaction term V as

$$\mathcal{L} = \psi^* K \psi - V(\psi \psi^*). \quad (3.11)$$

For a nonrelativistic theory one writes

$$K = \frac{\partial}{\partial \tau} - \frac{\vec{\nabla}^2}{2m} \quad (3.12)$$

whereas in the relativistic case one writes

$$K = \frac{\partial^2}{\partial \tau^2} - \vec{\nabla}^2. \quad (3.13)$$

The interaction term will be taken to be the one defined in (3.7) for the relativistic or the non-relativistic case. The relativistic example will be introduced only as an example of symmetry restoration. The spectrum, in the relativistic case, on both phases is known and will be a guide to the symmetry restored phase for the Hamiltonian (3.7) in the Helium-4 case.

From (3.10) one gets the free energy which is given by

$$F(T, V) = -\beta^{-1} \ln Z. \quad (3.14)$$

The whole thermodynamics can be obtained from the free energy.

III.3. Bose Einstein Condensation

The relevance of Bose-Einstein condensation for understanding the properties of liquid He^4 was pointed out first by London⁽⁴⁾ who, besides proving Bose-Einstein condensation for an ideal gas, suggested that the peculiar phase transition that liquid helium undergoes at 2.18° be regarded as a Bose-Einstein condensation phenomenon. The temperature for which a finite fraction of all helium atoms will be assembled in the lowest energy state, in the ideal gas approximation, is very close the critical temperature of the λ -point. The hypothesis that He^4 , in the superfluid phase, has to do with Bose-Einstein condensation received a boost after the seminal work of Bogoliubov⁽⁶⁾ who showed that the "phonon" part of spectrum of excitations of He^4 follow directly from Bose-Einstein condensation of an imperfect gas.

Within the field theoretical context Bose-Einstein condensation is intimately related to the spontaneous breakdown $U(1)$ symmetry since in the broken symmetry phase the expectation of value of the scalar field ψ is different of zero

$$\langle \psi \rangle = \sqrt{\rho_c}. \quad (3.15)$$

Under condition (3.15) we say, on the other hand, that the fluid is a superfluid. Actually Anderson's definition⁽²⁰⁾ of a superfluid is that it is a fluid for which condition (3.15) is met. Other approaches to superfluidity are based essentially on properties of Green's functions analogous to (3.15)^(21, 22). The description of Bose-Einstein condensation is then crucial for the understanding of the superfluid phase of Helium-4.

In order to study Bose-Einstein condensation we separate out the uniform (in

space) field theoretical configuration. In this way one writes, in analogy with Bogoliubov's approach^(5, 6),

$$\psi = \psi_0 + \psi'. \quad (3.16)$$

By substituting (3.16) into (3.10) one can write the free energy as a function of the Bogoliubov condensate⁽⁵⁾, that is

$$F = F(T, V, \psi_0) = F(V, T, \rho_0). \quad (3.17)$$

Since the number of particles with zero-momentum is given by

$$N_0 = V \rho_0 = V \psi_0^* \psi_0, \quad (3.18)$$

the dependence of F on ψ_0 is equivalent to the dependence of F on N_0 . As suggested by Glassgold, Kaufman and Watson⁽²³⁾ the important contribution comes from the term that minimizes F that is

$$\left. \frac{\partial F}{\partial \rho_0} \right|_{\rho_c} = 0. \quad (3.19)$$

Equation (3.19) is the basic equation for the determination of the occupation of the $\vec{p} = 0$ state.

The system described by (3.7) exhibits two phases. In the non-condensed phase

$$\rho_c = 0 \quad (3.20)$$

whereas in the condensed phase

$$\rho_c = \rho_c \left[\frac{N}{V}, T \right] = \rho_c[\rho_{\text{He}}, T]. \quad (3.21)$$

The critical temperature is the one that distinguishes the two phases. ρ_c plays the role of the order parameter. One defines the critical temperature as the one for which,

$$\rho_c \left[\frac{N}{V}, T_c \right] = 0. \quad (3.22)$$

Within the field theory one can get all relevant informations on Bose-Einstein condensation phenomena from the field theoretical Green's functions in momentum space computed at zero-momenta⁽¹⁸⁾. By following ref. (5) one can write, in field theory

$$F(\rho_0, V, T) = V \Gamma(\rho_0, T) \quad (3.23)$$

where $\Gamma(\rho_0, T)$ is the generating functional of the one-particle irreducible Green's functions at zero momenta. That is

$$\Gamma(\rho_0, T) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!} \frac{1}{n!} \tilde{\Gamma}^{(n,m)}(0 \dots 0, 0 \dots 0, T) \rho_0^{\frac{m+n}{2}} \quad (3.24)$$

where $\tilde{\Gamma}^{(n,m)}$ stands for the Fourier transform of the one-particle irreducible Green's functions at zero momenta

$$\tilde{\Gamma}^{(n,m)}(0, \dots, 0, T) = \tilde{\Gamma}^{(n,m)}(p_1 \dots p_m) \Big|_{p_1 = p_2 = \dots = p_n = 0} \quad (3.25)$$

As far as condensation phenomena is concerned $\Gamma(\rho_0, T)$ is the relevant potential. The whole thermodynamics can be inferred from $\Gamma(\rho_0, T)$.

From $\Gamma(\rho_0, T)$ one gets the occupation of the $\vec{p} = 0$ (ρ_c) from the equation

$$\left. \frac{\partial}{\partial \rho_0} (\Gamma(\rho_0, T)) \right|_{\rho_0 = \rho_c} = 0 \quad (3.26)$$

whereas the pressure (P), entropy (S), internal energy (U) and specific heat (C_V) are obtained from the equations

$$P = -\Gamma[\rho_c, T] \quad (3.27)$$

$$S = V \left[-\frac{\partial \Gamma(\rho_0, T)}{\partial T} \right]_{\rho_0 = \rho_c} \quad (3.28)$$

$$U = V \left[\Gamma[\rho_0, T] - T \frac{\partial \Gamma(\rho_0, T)}{\partial T} \right]_{\rho_0 = \rho_c} \quad (3.29)$$

$$C_V = \frac{dU}{dT} \quad (3.30)$$

Equations (3.26) and (3.27)-(3.29) are the basic equations of condensation phenomena.

III.4. Equation of State and London's Relation

Although equations (3.27)-(3.29) depends on the thermodynamical potential Γ which can be computed explicitly only under certain approximations, as we shall see in the next Chapter, one can derive two relevant properties without the explicit knowledge of Γ . These properties do not depend on the dynamics. They follow only from the fact that there is Bose-Einstein condensation.

The first property that one expects in Helium superfluid, if such a phenomena is related to Bose-Einstein condensation, is that the state of superfluid Helium should be characterized by zero Gibbs energy. In fact, it follows from (3.27)-(3.29) that

$$U + PV = TS(T) \quad (3.31)$$

From (3.31) it follows that

$$G = 0 \quad (3.32)$$

As a result of the third law of thermodynamics one expects that

$$\lim_{T \rightarrow 0} TS(T) = 0 \quad (3.33)$$

By dividing (3.31) by N and then taking the zero temperature limit, we can predict the following equation of state

$$\frac{P}{\rho_{He}} = -\varepsilon \quad (3.34)$$

where ε is the binding energy per atom defined in (2.5).

The static fountain effect can be explained from London's relation (eq. (2.13)). We can see that this relation follows as a consequence of our basic equations describing the superfluid phase. In fact, if one derives the pressure as function of T , one gets (independently of the details of the dynamics), from (3.25)

$$\frac{dP}{dT} = \frac{\partial \Gamma(\rho_c, T)}{\partial \rho_c} \frac{d\rho_c}{dT} - \left(\frac{\partial \Gamma(\rho_0, T)}{\partial T} \right)_{\rho_0 = \rho_c}$$

From equations (3.26) and (3.28) it follows that

$$dP = \frac{S}{V} dT \quad (3.35)$$

That is, we get London's relation. It follows from the equilibrium thermodynamics when Bose-Einstein condensation occurs. London's relation follows also from (3.32) since

$$dG = P dV - S dT \quad (3.36)$$

IV. ONE LOOP APPROXIMATION: GENERAL RESULTS

IV.1. Loop Expansion

The loop expansion provides a systematic as well as a proper method for dealing with Bose-Einstein condensation. As pointed out in ref. (5) the method is richer and more appropriate than the perturbative approach on the low density approximation.

Within the loop expansion one can write the effective potential as a series

$$\Gamma(\rho_0, T) = \sum_{n=0}^{\infty} \Gamma^{(n)}(\rho_0, T) \quad (4.1)$$

where in $\Gamma^{(n)}$ one considers only those irreducible Green's function having n -loops.

At the classical level (zero-loop approximation) one can write⁽⁵⁾, for model (3.7),

$$\Gamma^{(0)}(\rho_0, T) = -\Delta E \rho_0 + \frac{\lambda}{2} \rho_0^2 \quad (4.2)$$

One can compute Γ explicitly up to one loop level. The answer, for our model, can be inferred from Toyoda's papers⁽²⁴⁾. However, in order to understand the roton spectrum, it is better to present an alternative derivation of Toyoda's result.

IV.2. One Loop Approximation

Up to the one loop approximation one can write^(16,17)

$$\begin{aligned} F &= F_0 + F_1 \\ &= V [\Gamma^{(0)} + \Gamma^{(1)}] \end{aligned} \quad (4.3)$$

where F_0 is the zero loop approximation contribution from (4.2) given by

$$F_0(\rho_0, V, T) \equiv V \left[-(\Delta E) \rho_0 + \frac{\lambda}{2} \rho_0^2 \right] \quad (4.4)$$

$\Gamma^{(1)}$, in (4.3), represents the first quantum corrections and after substituting (3.16) into (3.10) with the interaction given by (3.7) and then expanding in ψ' taking into account only quadratic terms, we end up with a quadratic functional integral in ψ' . The result is

$$\begin{aligned} \Gamma^{(1)}(\rho_0, T) &= \int \mathcal{D}\psi' \mathcal{D}\psi'^* \exp -\frac{1}{2} \int (\psi'^*, \psi') G \begin{pmatrix} \psi' \\ \psi'^* \end{pmatrix} \\ &\equiv \det G \end{aligned} \quad (4.5)$$

Where G is a matrix element which in momentum space have matrix elements given by

$$G(n, \vec{k}, T, \rho_0) = \begin{bmatrix} \bar{K}(n, \vec{k}, \beta) - \Delta E + 2\lambda\rho_0 & \lambda(\psi_0^*)^2 \\ \lambda(\psi_0^2) & \bar{K}^*(n, \vec{k}, \beta) - \Delta E + 2\lambda\rho_0 \end{bmatrix} \quad (4.6)$$

where \bar{K} is the Fourier transform of the kinetic term K . In the non-relativistic case \bar{K} is given by

$$\bar{K} = \frac{2in\pi}{\beta} + \frac{\vec{k}^2}{2m} \equiv i\omega_n + \frac{\vec{k}^2}{2m} \quad (4.7)$$

and in the relativistic case, one has

$$\bar{K}(n, \vec{k}, \beta) = \left(\frac{2n\pi}{\beta}\right)^2 + \vec{k}^2 \quad (4.8)$$

One can diagonalize the matrix G by using a Bogoliubov's transformation and write

$$G \rightarrow G' = \begin{pmatrix} \lambda^{(+)}(\vec{k}, \rho_0, \beta) & 0 \\ 0 & \lambda^{(-)}(\vec{k}, \rho_0, \beta) \end{pmatrix} \quad (4.9)$$

where the eigenvalues λ^\pm are given by

$$\lambda^\pm = \text{Re } \bar{K}(n, \vec{k}, \beta) - \Delta E + 2\lambda\rho_0 \pm \sqrt{\lambda^2\rho_0^2 - (\text{Im } \bar{K}(k, n, \beta))^2} \quad (4.10)$$

with $\lambda^{(\pm)}$ given by (4.10) one can write, up to the one loop level,

$$\frac{F(\rho_0, V, T)}{V} = -(\Delta E)\rho_0 + \frac{\lambda}{2}\rho_0^2 + \frac{1}{2\beta} \sum_n \int \frac{d^3\vec{k}}{(2\pi)^3} \ln \lambda^{(+)} \lambda^{(-)} \quad (4.11)$$

This expression is the basic one in order to study superfluid Helium at the one loop level.

In the non-relativistic case

$$\lambda^+ \lambda^- = \left(\frac{2n\pi}{\beta}\right)^2 + \left[\left(\frac{\vec{p}^2}{2m}\right) - \Delta E + 2\lambda\rho_0\right]^2 - \lambda^2\rho_0^2 \quad (4.12)$$

whereas in the relativistic case

$$\lambda^+ \lambda^- = \left[\vec{p}^2 - \Delta E + 2\lambda\rho_0 + \left(\frac{2n\pi}{\beta}\right)^2\right]^2 - \lambda^2\rho_0^2 \quad (4.13)$$

in both cases we can perform the sum over n by using the identity

$$\frac{1}{\beta} \sum_n \ln \left(\left(\frac{2n\pi}{\beta}\right)^2 + \omega^2 \right) = \omega + \frac{2}{\beta} \ln \left(1 - e^{-\beta\omega} \right) \quad (4.14)$$

The general structure of F up to the one loop approximation is, by using (4.14),

$$\begin{aligned} F(\rho_0, V, T) &= V \left[-\Delta E\rho_0 + \frac{\lambda}{2}\rho_0^2 + \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\omega(\vec{k}, \rho_0)}{2} \right] \\ &\quad + \frac{V}{\beta} \int \frac{d^3\vec{k}}{(2\pi)^3} \ln \left(1 - e^{-\beta\omega(\vec{k}, \rho_0)} \right) \end{aligned} \quad (4.15)$$

where in (4.15) we have separated the zero temperature contribution (the temperature independent part in (4.15)) from the temperature dependent contribution. ω in (4.15) can be inferred from (4.12) and (4.13).

The zero temperature contribution is divergent and requires renormalization. Up to the one loop approximation one has to add the usual counterterms to the Hamiltonian (7). One has also to remove a zero point energy contribution in (4.15). The equation for the occupation of the $\vec{p}=0$ is

$$\Delta E = \lambda\rho_c + \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{d\omega(\vec{k}, \rho_0)}{d\rho_0} \Big|_{\rho_0=\rho_c} + \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\frac{d\omega}{d\rho_0}}{e^{\beta\omega} - 1} \Big|_{\rho_0=\rho_c} \quad (4.16)$$

The equation (4.16) is also divergent but these divergences can be eliminated by the usual procedure of adding counterterms^(5, 20).

After renormalizing (4.16) we could in principle discuss the depletion of the condensate as a result of quantum effects⁽²⁵⁾.

In this paper we will work within the semiclassical approach for computing the thermodynamics of He-4. Within the semiclassical method all one has to do is a zero-point energy subtraction (zero point energy of phonons).

IV.3. Relativistic Case

In the relativistic case one can consider two cases. In the first case

$$-\Delta E < 0 \quad (4.17)$$

and the theory exhibits spontaneous breakdown of $U(1)$ symmetry. The condition

$$\frac{dF}{d\rho_0} \Big|_{\rho_c} = 0 \quad (4.18)$$

determines the occupation of the $\vec{p}=0$ states as a function of the temperature. At

the classical level one has

$$\rho_c^{(0)} = \frac{\Delta E}{\lambda} \quad (4.19)$$

Since in the relativistic case

$$\lambda^+(\vec{k}, n, \rho_0) = \left(\frac{2\pi n}{\beta}\right)^2 + \vec{k}^2 - \Delta E + 3\lambda\rho_0 \quad (4.20)$$

$$\lambda^-(\vec{k}, n, \rho_0) = \left(\frac{2\pi n}{\beta}\right)^2 + \vec{k}^2 - \Delta E + \lambda\rho_0$$

The spectrum of excitation involves two types of particles of masses M_1 and M_2 , whose masses are given by the square root of $\lambda^\pm(n=0, \vec{k}=0)$. One gets

$$M_1^{BS} = \lambda^+(n=0, \vec{k}=0, \rho_c) = \sqrt{2\Delta E} \quad (4.21)$$

$$M_2^{BS} = \lambda^-(n=0, \vec{k}=0, \rho_c) = 0$$

that is, one gets a massive particle and one massless Goldstone boson⁽²⁶⁾.

The thermodynamics of the system at very low temperatures will be entirely controlled by the Goldstone boson mode. One gets explicitly in this phase

$$\frac{1}{V} F(\rho_0, T) = -\Delta E\rho_0 + \frac{\lambda\rho_0^2}{2} + \frac{1}{2\beta} \sum \int \frac{d^3\vec{k}}{(2\pi)^3} \left[\ln \left(\left(\frac{2n\pi}{\beta}\right)^2 + \vec{k}^2 + 2\Delta E \right) + \ln \left(\left(\frac{2n\pi}{\beta}\right)^2 + \vec{k}^2 \right) \right] \quad (4.22)$$

In the symmetry restored phase, characterized by

$$-\Delta E > 0 \quad (4.23)$$

the spectrum is also well known. There are two particles with the same mass $\sqrt{\Delta E}$

$$M_1 = M_2 = \sqrt{\Delta E} \quad (4.24)$$

These particles obey the dispersion relation

$$\omega(\vec{p}) = \sqrt{\vec{p}^2 + \Delta E} \quad (4.25)$$

The spectrum for the symmetric phase ($\Delta E > 0$) can be obtained directly from the broken symmetry phase if one makes use of a simple approach to symmetry restoration. This will be explained in the next section.

IV.4. Simple Approach to Symmetry Restoration

We shall start now from the broken symmetry phase in which the matrix G is diagonalized and write

$$G' = \begin{bmatrix} \left(\frac{2n\pi}{\beta}\right)^2 + \vec{k}^2 + (\sqrt{2\Delta E})^2 & 0 \\ 0 & \left(\frac{2n\pi}{\beta}\right)^2 + \vec{k}^2 \end{bmatrix} \quad (4.26)$$

Let us now introduce a coupling between the massive particles of mass $\sqrt{2\Delta E}$ and the Goldstone mode of the form

$$G' \rightarrow G'' = \begin{bmatrix} \left(\frac{2n\pi}{\beta}\right)^2 + \vec{k}^2 + 2\Delta E & i\Delta E \\ i\Delta E & \left(\frac{2n\pi}{\beta}\right)^2 + \vec{k}^2 \end{bmatrix} \quad (4.27)$$

The new eigenvalues are

$$\lambda^+(n, \vec{k}) = \lambda^-(n, \vec{k}) = \left(\frac{2n\pi}{\beta}\right)^2 + \vec{k}^2 + \Delta E \quad (4.28)$$

This is result (4.25).

The spectrum of the symmetric phase is thus obtained as a result of the coupling (4.27) between the two modes (4.21).

V. HELIUM SUPERFLUID AT VERY LOW TEMPERATURE

Let us show that model (3.7) describes He^4 at very low temperatures. Since the system is a nonrelativistic one can write, by using (4.12), the free energy (4.11) as

$$F(\rho_0, V, T) = V \left[-\Delta E \rho_0 + \frac{\lambda \rho_0^2}{2} \right] + \frac{1}{2\beta} \sum_n \int \frac{d^3 \vec{k}}{(2\pi)^3} \ln \left[\left(\frac{2n\pi}{\beta} \right)^2 + \omega^2[\vec{k}, \rho_0] \right] \quad (5.1)$$

where

$$\omega^2[\vec{k}, \rho_0] = \left(\frac{\vec{k}^2}{2m} - \Delta E + 2\lambda \rho_0 \right)^2 - \lambda^2 \rho_0^2 \quad (5.2)$$

After subtracting the zero point energy contribution one gets

$$F(\rho_0, V, T) = V \left[-\Delta E \rho_0 + \frac{\lambda \rho_0^2}{2} \right] + \frac{V}{\beta} \int \frac{d^3 \vec{k}}{(2\pi)^3} \ln \left(1 - e^{-\beta \omega(\vec{k}, \rho_0)} \right) \quad (5.3)$$

The occupation of the $\vec{p} = 0$ (ρ_c) should be inferred from the minimization of the free energy (5.3). One gets, from (5.3) and (4.14)

$$\lambda \rho_c = \Delta E - \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{e^{\beta \omega(\vec{k}, \rho_0)} - 1} \left. \frac{d\omega}{d\rho_0} \right|_{\rho_0 = \rho_c} \quad (5.4)$$

Equation (5.4) allows us to get the occupation of the $\vec{p} = 0$ state as a function of the temperature within the one-loop approximation. $\omega(\vec{k}, \rho_0)$ in (5.4) is a function of the ρ_0 and given by (5.2). Once ρ_c is obtained, from (5.4), one can get, by using (3.27)–(3.29) other thermodynamical variables. The pressure as a function of the condensate, up to the one loop level, is given by

$$P = +\Delta E \rho_c - \frac{\lambda \rho_c^2}{2} - \frac{1}{\beta} \int \frac{d^3 \vec{k}}{(2\pi)^3} \ln \left(1 - e^{\beta \omega(\vec{k}, \rho_c)} \right) \quad (5.5)$$

The entropy, on the other hand from (3.28) and (5.3), is

$$\frac{S}{V} = - \int \frac{d^3 \vec{k}}{(2\pi)^3} \ln \left(1 - e^{-\beta \omega(\vec{k}, \rho_c)} \right) + \beta \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\omega(\vec{k}, \rho_c)}{e^{\beta \omega(\vec{k}, \rho_c)} - 1} \quad (5.6)$$

whereas the internal energy will be

$$U = V \left[-\Delta E \rho_c + \frac{\lambda \rho_c^2}{2} \right] + \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\omega(\vec{k}, \rho_c)}{e^{\beta \omega(\vec{k}, \rho_c)} - 1} \quad (5.7)$$

and

$$C_V = \beta^2 \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\omega^2(\vec{k}, \rho_c)}{e^{\beta \omega(\vec{k}, \rho_c)} - 1} \quad (5.8)$$

In order to get the thermodynamics at very low temperatures let us solve equation (5.4) in the zero temperature limit.

At zero temperatures the solution to (5.4) will be given by the classical solution

$$\rho_c = \frac{\Delta E}{\lambda} = \frac{m \Delta E}{8\pi a} \quad (5.9)$$

In this limit the condensate contains, in the dilute gas approximation, all the particles. In this way one gets the Helium density from (5.9). That is

$$\rho_{\text{He}} = \frac{m \Delta E}{8\pi a} \quad (5.10)$$

One can express the Helium density in terms of the three relevant parameters obtaining a result that, as shown in Appendix A, is very close to the experimental value given in (2.7).

By using (5.9) as an approximate solution to (5.4) one gets the following results, valid in the low temperature limit,

$$\frac{U}{V} = \left[-\frac{\Delta E m}{16\pi a} \right] + \frac{\pi^2}{30} \left(\sqrt{\frac{m}{\Delta E}} \right)^3 T^4 \quad (5.11)$$

$$s \equiv \frac{S}{V} = \frac{2\pi^2}{45} \left(\sqrt{\frac{m}{\Delta E}} T \right)^3 \quad (5.12)$$

$$P = \frac{m \Delta E^2}{16\pi a} + \frac{\pi^2}{90} \left(\sqrt{\frac{m}{\Delta E}} \right)^3 T^4 \quad (5.13)$$

$$\frac{C_V}{V} = \frac{4\pi^2}{30} \left(\sqrt{\frac{m}{\Delta E}} \right)^3 T^3 \quad (5.14)$$

If we substitute the values of the constants m , ΔE , and a given by (1.2) one gets the following predictions

$$\frac{U}{V} = -7 \cdot 10^4 (\text{eV})^4 + \frac{\pi^2}{30} 8 \cdot 10^{18} T^4 \quad (5.15)$$

$$\frac{S}{V} = \frac{2\pi^2}{45} 8 \cdot 10^{18} T^3 \quad (5.16)$$

$$P = 7 \cdot 10^4 (\text{eV})^4 + \frac{\pi^2}{90} 8 \cdot 10^{18} T^4 \quad (5.17)$$

$$C_V = \frac{4\pi^2}{30} 8 \cdot 10^{18} T^3 \quad (5.18)$$

C_V and the entropy have a temperature dependence that is well known theoretically and verified experimentally. Let us turn then to the internal energy and pressure. First of all one can relate these quantities as follows

$$V(P(T) - P(0)) = \frac{1}{3}(U(T) + U(0)) = \frac{1}{4} T S(T) \quad (5.19)$$

where $U(0)$ is the internal energy density at zero temperature that, from (5.7) and (5.9), is given by

$$U(0) = -\frac{\Delta E m}{16\pi a} = -\frac{\lambda}{2} \rho_{\text{He}}^2 \quad (5.20)$$

And $P(0)$ is the zero temperature pressure, which, from (5.5), (5.13) and (5.9) is

$$P(0) = \frac{\Delta E m}{16\pi a} = \frac{\lambda}{2} \rho_{\text{He}}^2 \quad (5.21)$$

For temperatures satisfying $T < 2^\circ\text{K}$ then, from (5.15) and (5.17) it follows that the interval energy and pressure are approximately constant and given by (5.20) and (5.21).

From (5.15)–(5.17) one can check the validity of equation (3.31) up to one loop approximation.

At low temperatures one can predict the following equation of state

$$PV = -U(0) = \text{constant} \quad (5.22)$$

This is, up to the one loop approximation, equation (3.34).

One can predict, from (5.20), a binding energy per atom defined as

$$\varepsilon \equiv \frac{U}{N} \quad (5.23)$$

obtaining, from (5.7)

$$\varepsilon = -\frac{\Delta E}{2} = 5.1 \cdot 10^{-4} \text{ eV} \quad (5.24)$$

Our prediction (5.24) is very close to the experimental value (2.6).

Furthermore, one gets a surprising relation among the physically measurable quantities $P(0)$, ε and ρ_{He} . This relation is a consequence of the equation of the state (5.22)

$$\frac{P(0)}{\rho_{\text{He}}} = -\varepsilon \quad (5.25)$$

Finally, if one differentiates (5.19) with respect to T and using (5.16)–(5.17) we will get up to one loop approximation, London's relation

$$dP = s(T) dT \quad (5.26)$$

At the one loop level one can see explicitly that

$$PV + U - ST = 0$$

as predicted by (3.31).

SOUND SPEED

The sound speed, defined by,

$$v_s = \sqrt{\frac{\partial P}{\partial \rho}} \quad (5.27)$$

can be obtained from (5.20). One gets

$$v_s = \sqrt{\frac{\lambda \rho_{\text{He}}}{m}} = \sqrt{\frac{\Delta E}{m}} \quad (5.28)$$

By using the values of ΔE and m in (1.2) we get

$$v_s = 5 \cdot 10^{-7} \quad (5.29)$$

PHONON SPECTRUM

For ρ_c given by (5.4) one gets, from (5.2) the dispersion relation

$$\omega(\vec{p}) = \sqrt{\vec{p}^2 \frac{\Delta E}{m} + \left(\frac{\vec{p}^2}{2m}\right)^2} \quad (5.30)$$

For small momenta, one finds, from (5.30) the phonon spectrum

$$\omega^{\text{ph}}(\vec{p}) = p \sqrt{\frac{\Delta E}{m}} \quad (5.31)$$

which, by using (5.28) can be written as usual, as:

$$\omega^{\text{ph}}(\vec{p}) = v_s p \quad (5.32)$$

ROTON SPECTRUM

The effective Hamiltonian (3.18) describes Helium-4 at very low temperatures. As the temperature increases other effective Hamiltonian might be more appropriate.

However if we just admit that what happens at higher temperatures is symmetry restoration then, by following our strategy for the relativistic case, one can get the relevant part of the spectrum at high temperatures. We just recall that in the relativistic case one can get the spectrum of excitations in the symmetric phase by introducing a coupling between the low temperature phase excitations λ^+ and λ^- . By repeating this strategy in the nonrelativistic case one introduces a coupling between the Goldstone mode and the massive mode through the substitution

$$G \rightarrow G' = \begin{pmatrix} \lambda^+(\rho_0, n, \vec{p}) & i\Delta E \\ i\Delta E & \lambda^-(\rho_0, n, \vec{p}) \end{pmatrix}. \quad (5.33)$$

The coupling between the Goldstone mode and the massive one leads to the following new modes

$$\begin{aligned} \lambda^+(n, \vec{p}, \rho_0) &= \frac{\vec{p}^2}{2m} - \Delta E + 2\lambda\rho_0 + \sqrt{\lambda^2\rho_0^2 - \left(\frac{2n\pi}{\beta}\right)^2 - (\Delta E)^2} \\ \lambda^-(n, \vec{p}, \rho_0) &= \frac{\vec{p}^2}{2m} - \Delta E + 2\lambda\rho_0 - \sqrt{\lambda^2\rho_0^2 - \left(\frac{2n\pi}{\beta}\right)^2 - (\Delta E)^2} \end{aligned} \quad (5.34)$$

and consequently $F \rightarrow F'$ where

$$F'(\rho_0, V, T) = V \left[-\Delta E\rho_0 + \frac{\Delta\rho_0^2}{2} \right] + \frac{V}{\beta} \int \frac{d^3\vec{k}}{(2\pi)^3} \ln \left(1 - e^{-\beta\omega(\vec{k}, \rho_0)} \right) \quad (5.35)$$

and the new dispersion relation is

$$\omega(\vec{k}, \rho_0) = \sqrt{\left(\frac{\vec{k}^2}{2m} - \Delta E + 2\lambda\rho_0\right)^2 + (\Delta E)^2 - \lambda^2\rho_0^2}. \quad (5.36)$$

In order to get the thermodynamical properties of the system, and as can be inferred from (5.5)–(5.8), all one has to do is to substitute ρ_0 by ρ_c given by (5.4). In this way, for phenomenological applications one has to analyse

$$\omega(\vec{k}, \rho_c) \equiv \sqrt{\left(\frac{\vec{k}^2}{2m} - \Delta E + 2\lambda\rho_c\right)^2 + (\Delta E)^2 - \lambda^2\rho_c^2}. \quad (5.37)$$

The main prediction that can be made by using (5.37) is that the spectrum exhibits different features as the condensate becomes less populated. In particular, the spectrum (5.37) has a minimum whenever

$$\rho_c(T) < \frac{1}{2} \rho_c(T=0) \quad (2\lambda\rho_c < \Delta E). \quad (5.38)$$

When condition (5.38) is met, then, from (5.37) it follows a roton-like dispersion relation:

$$\omega(\vec{k}) \cong \sqrt{\left(\frac{\vec{k}^2}{2m} - \Delta E\right)^2 + (\Delta E)^2}. \quad (5.39)$$

For comparison we write the phonon spectrum

$$\omega^{\text{phonon}}(\vec{k}) = \sqrt{\left(\frac{\vec{k}^2}{2m} + \Delta E\right)^2 - (\Delta E)^2}. \quad (5.40)$$

The spectrum (5.37) has a minimum for

$$p_0 = \sqrt{2m\Delta E}. \quad (5.41)$$

At the minimum we get

$$\Delta = E(\vec{p}_0) = \Delta E. \quad (5.42)$$

The first non-trivial relation that follows from (5.41) and (5.42) is that

$$\Delta \equiv \frac{\vec{p}_0^2}{2m}. \quad (5.43)$$

Finally by expanding around $p = p_0$ one gets the roton spectrum

$$\omega^{\text{roton}}(p) = \frac{(p - p_0)^2}{m} + \Delta. \quad (5.44)$$

From (5.44) and (2.2) it follows that

$$\mu = \frac{1}{2} m \quad (5.45)$$

In this way we have been successful in getting Landau's roton spectrum. Landau's parameters p_0 , Δ and μ are predicted by our expressions (5.41), (5.42) and (5.45), by using (1.2) to be

$$\begin{aligned} \Delta &= 10.2 \cdot 10^4 \text{ eV} & \Delta_{\text{exp}} &= 9 \cdot 10^4 \text{ eV} \\ p_0 &= 2.8 \cdot 10^3 \text{ eV} & p_{0\text{exp}} &= 3.8 \cdot 10^3 \text{ eV} \\ \mu &\cong 1.75 \cdot 10^9 \text{ eV} & 1.12 \cdot 10^9 \text{ eV} &< \mu_{\text{exp}} < 2.62 \cdot 10^9 \text{ eV} \end{aligned} \quad (5.46)$$

The result for the gap is in an excellent agreement with the experimental result. The result for p_0 is not so good. The experimental result for μ seems to depend on the method for its determination (calorimetric method is the upper bound and neutron scattering corresponds to the lower bound)⁽²⁷⁾. In ref. (9) it has been pointed out that both methods seems to converge to a value smaller than the lower bound.

SUPERFLOW SPEED LIMIT

Our dispersion relation (5.37) can be written as

$$\omega(\vec{p}) = \sqrt{\left(\frac{p^2}{2m} - \frac{p_0^2(T)}{2m}\right)^2 + \Delta^2(T)} \quad (5.47)$$

where

$$p_0^2(T) = 2m[(\Delta E) - 2\lambda \rho_c(T)] \quad (5.48)$$

$$\Delta^2(T) = (\Delta E)^2 - \lambda^2 \rho_c(T) \quad (5.49)$$

The superflow speed limit, should be given, from (5.47) and (2.10) as

$$v < \left(\frac{1}{m} \left[\sqrt{\left(\frac{p_0^2(T)}{2m} + \Delta^2(T)\right)^2} - \frac{p_0^2(T)}{2m} \right] \right)^{1/2} \quad (5.50)$$

Expression (5.50) allows us to get an upper bound for the critical velocity. This maximum speed is, from (5.50)

$$v < \sqrt{\frac{\max \Delta(T)}{m}} = \sqrt{\frac{\Delta E}{m}} = v_s \quad (5.51)$$

The sound speed is then an upper bound to the critical velocities. It follows also from (5.51) that, if

$$\Delta(T) = 0 \quad (5.52)$$

then, there is no superfluidity.

We have seen in this Chapter that most features of superfluid Helium-4 can be described in a very simple way. We have made a prediction on the zero-temperature equation which deserves to be tested. Our prediction for $P(0)$ implies

$$P(0) \cong 6 \cdot 10^4 (\text{eV})^4 \quad (5.53)$$

By extrapolating the isopycnals in Keesom's book⁽²⁸⁾ we would get for $\rho = 1.6 \cdot 10 (\text{eV})^3$ a value of

$$P(0) \cong 4.8 \cdot 10^4 (\text{eV})^4 \quad (5.54)$$

By using (5.54) and (2.7) we will get

$$\frac{P(0)}{\rho_{\text{He}}} = 3 \cdot 10^{-4} \text{ eV} \quad (5.55)$$

Our model predicts, from (5.54) and (5.10)

$$\frac{P(0)}{\rho_{\text{He}}} = 5 \cdot 10^{-4} \text{ eV} \quad (5.56)$$

VI. NORMAL AND SUPERFLUID COMPONENTS

VI.1. Two-Fluid Picture

In Chapter II we have pointed out that superfluid Helium behaves as a mixture of two-fluids. The nature of these fluid components has not yet been settled⁽¹⁴⁾. By looking at our dispersion relation it is possible to shed some light on this problem. First of all one can see that for

$$p \gtrsim p_0 = \sqrt{2m\Delta} \quad (6.1)$$

the spectrum of the excitations can be approximated by

$$\omega^r(p) = \frac{(p-p_0)^2}{2\mu} + \Delta \quad (6.2)$$

However for excitations such that

$$p > p_0 \quad (6.3)$$

The dispersion relation is of the form

$$\omega^n(p) = \frac{\vec{p}^2}{2m} \quad (6.4)$$

In this way one expects two types of excitations. Those whose momenta obeys (6.1) are rotons. Those obeying (6.3) will be named normal. For very low temperatures one can also distinguish these two types of excitations (although as we will see the normal component is practically absent). The first excitation will be the phonon

$$\omega(\vec{p}) = v_s |p| \quad (6.5)$$

For momenta such that

$$p > p_0 \quad (6.6)$$

One has

$$\omega^n(\vec{p}) = \frac{\vec{p}^2}{2m} \quad (6.7)$$

From (6.2), (6.4), (6.5) and (6.7) it follows that one has two types of excitations in the system. Normal excitations are those with low momentum, that is, those satisfying (6.4). The other excitations can be named superfluid.

From the previous distinction one can separate each physical quantities in momentum space and write, whenever integral over momenta occurs

$$\int d^3\vec{p} = \int_{\text{normal}} d^3\vec{p} + \int_{\text{superfluid}} d^3\vec{p} \quad (6.8)$$

where, from (6.3) or (6.6),

$$\int_{\text{normal}} d^3\vec{p} \equiv \int_{p > p_0} d^3\vec{p} \quad (6.9)$$

and the superfluid component is just the complement of (6.9) and should be inferred from (6.9) and (6.8).

From (6.2), (6.4), (6.5) and (6.7) we will be able to define approximations for the integrand of certain integrals. For example, if one defines

$$N(\vec{p}) \equiv \left(\exp\left(\frac{\omega(\vec{p})}{T}\right) - 1 \right)^{-1} \quad (6.10)$$

Then, from (6.2)-(6.7) and (5.39) and (5.40) one can define

$$N^{(n)}(\vec{p}) = \left(\exp\left(\frac{\vec{p}^2}{2mT}\right) - 1 \right)^{-1} \quad (6.11)$$

$$N^{(r)}(\vec{p}) = \left(\exp\left(\frac{(p-p_0)^2}{2\mu T} + \frac{\Delta}{T}\right) - 1 \right)^{-1} \quad (6.12)$$

$$N^{(p)}(\vec{p}) = \left(\exp\left(\frac{v_s \vec{p}}{T}\right) - 1 \right) \quad (6.13)$$

From (6.11)-(6.13) one can define normal, roton and phonon thermal averages. For the kinetic energy one can define the roton contribution as

$$\langle K \rangle_r = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{\vec{p}^2}{2m} N^{(r)}(\vec{p}) \quad (6.14)$$

Expression (6.14) is just, in Landau's notation $(\rho^{(n)})_r$, that is

$$(\rho^{(n)})_r = \frac{2m}{3T} \langle K \rangle_r \quad (6.15)$$

whereas the average number of rotons is given by

$$\langle N \rangle_r = \int \frac{d^3 \vec{p}}{(2\pi)^3} N^{(r)}(\vec{p}) \quad (6.16)$$

VI.2. Normal Component of the Fluid Density

Our identification of the normal and superfluid excitation allows us to write explicit expressions for the normal and superfluid component of the fluid density. In order to do this we write the basic equation of condensation phenomena explicitly

$$-\Delta E + \lambda \rho_c + \frac{2\lambda}{(2\pi)^3} \int d^3 \vec{p} \frac{\left\{ \left(\frac{\vec{p}^2}{2m} \right) - \Delta E + 2\lambda - \frac{\lambda \rho_c}{2} \right\}}{\sqrt{\left(\frac{\vec{p}^2}{2m} - \Delta E + 2\lambda \rho_c \right)^2 + (\Delta E)^2 + (\lambda \rho_c)^2}} \otimes$$

$$\otimes \frac{1}{\left[\exp \beta \sqrt{\left(\frac{\vec{p}^2}{2m} - \Delta E + 2\lambda \rho_c \right)^2 + (\Delta E)^2 - (\lambda \rho_c)^2} - 1 \right]} = 0 \quad (6.17)$$

If one divides (6.17) by λ and by using (5.9) and (5.10) and neglecting the condensate dependence in the integrals in (6.17) one can write

$$\rho_{\text{He}} = \rho_c + \frac{2}{(2\pi)^3} \int d^3 \vec{p} \frac{\left(\frac{\vec{p}^2}{2m} - \Delta E \right)}{\sqrt{\left(\frac{\vec{p}^2}{2m} - \Delta E \right)^2 + \Delta E^2}} \frac{1}{\left[\exp \beta \sqrt{\left(\frac{\vec{p}^2}{2m} - \Delta E \right)^2 + \Delta E^2} - 1 \right]} \quad (6.18)$$

Let us decompose the integral in (6.18) into two pieces, following (6.8), and write

$$\rho_{\text{He}} = \rho_s + \rho_n \quad (6.19)$$

The normal component of the Helium density should be inferred from (6.17) when one takes into account restriction (6.6). For p satisfying (6.6) the integrand of (6.18) can be approximated by

$$\frac{\frac{\vec{p}^2}{2m} - \Delta E}{\sqrt{\left(\frac{\vec{p}^2}{2m} - \Delta E \right)^2 + \Delta E^2}} \frac{1}{\exp \left(\frac{\varepsilon(\vec{p})}{T} \right) - 1} \quad p^2 > p_0^2 \quad \frac{1}{\exp \left(\frac{\vec{p}^2}{2mT} \right) - 1} \quad (6.20)$$

The normal component of the superfluid density can be written, from (6.20) and (6.9) as

$$\rho_n(T) = \frac{1}{(2\pi)^3} \int_{p > p_0} \frac{d^3 \vec{p}}{\exp \left(\frac{\vec{p}^2}{2m} \frac{1}{T} \right) - 1} \equiv \int_{p > p_0} \frac{d^3 \vec{p}}{(2\pi)^3} N^{(n)}(\vec{p}) \quad (6.21)$$

If one takes for p_0 the lower limit of (6.21) one can write a simple analytical expression for the normal component. This expression is

$$\rho_n(T, \Delta E) = (2mT)^{3/2} h_0 \left(\frac{\Delta E}{T} \right) \quad (6.22)$$

where

$$h_0 \left(\frac{\Delta E}{T} \right) = \frac{1}{\pi^2} \int_{\sqrt{\Delta E/T}}^{\infty} dy y^2 \frac{1}{e^{y^2} - 1} \quad (6.23)$$

In fig. (2) we show the behavior of the function

$$\frac{\rho_n(T, \Delta E)}{\rho_n(T_\lambda, \Delta E)} \equiv \left(\frac{T}{T_\lambda} \right)^{3/2} \frac{h_0 \left(\frac{\Delta E}{T} \right)}{h_0 \left(\frac{\Delta E}{T_\lambda} \right)} \quad (6.24)$$

From fig. (2) we can see that our expression (6.24) gives an strikingly good result from the experimental point of view. In fig. (2) we present also the result of Andronikashvili⁽²⁹⁾. The agreement is very good. We have verified that an even better result is obtained if one uses instead of (6.21)

$$\rho_n(T, \Delta E) = 2 \int_{p > p_0} \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\exp\left(\frac{\omega(\vec{p})}{T}\right) - 1} \quad (6.25)$$

with $\omega(\vec{p})$ given by (5.39).

One can easily see that ρ_n defined in (6.19) exhibits the following properties

$$\rho_n \xrightarrow{T \rightarrow 0} 0 \quad (6.26)$$

$$\rho_n \xrightarrow{T \rightarrow \infty} 2\rho_{\text{gas}}(T) \equiv 2 \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\left[\frac{\vec{p}^2}{2mT} - 1\right]} \quad (6.27)$$

where ρ_{gas} is the free gas density. In particular, from (6.22) and (6.23), that for $\Delta E = 0$

$$\rho_n(\Delta E = 0, T) = 2\rho_{\text{gas}}(T) \quad (6.28)$$

VI.3. Superfluid Component of the Fluid Density

The definition of the superfluid component is, from (6.19), given by

$$\rho_s = \rho_{\text{He}} - \rho_n \quad (6.29)$$

where ρ_n is defined by (6.21).

As can be seen from (6.18) part of the superfluid component of the fluid density is made of particles that are in the condensed state (the $\vec{p} = 0$ state). The other part has a contribution due to excitations. In this way one has, on general grounds

$$\rho_s \cong \rho_c(T) + \rho_{\text{exc}}(T) \quad (6.30)$$

The contribution due to excitation depends on whether phonons or roton contributions are the relevant excitations. In the following we make a simple estimate of the roton contribution to the superfluid component as defined by equation (6.29). For the roton spectrum the relevant integration comes from the region in momentum space for which $p \cong p_0$. Let us then define from (6.18) the roton contribution to ρ_s as

$$(\rho_s)_{\text{roton}} = 4\pi \int_0^{2p_0} \frac{p^2 dp}{(2\pi)^3} \frac{\left(\frac{\vec{p}^2}{2m} - \Delta E\right)}{\sqrt{\left(\frac{\vec{p}^2}{2m} - \Delta E\right)^2 + \Delta E^2}} \frac{1}{\left[\exp \beta \sqrt{\left(\frac{\vec{p}^2}{2m} - \Delta E\right)^2 + \Delta E^2} - 1\right]} \quad (6.31)$$

After a change of variables (6.31) can be written under the alternative form

$$\begin{aligned} (\rho_s)_{\text{rotons}} &= \frac{1}{\pi^2} p_0^3 \int_0^2 \frac{y^2 dy (y^2 - 1)}{\sqrt{(y^2 - 1)^2 + 1}} \frac{1}{\exp\left(\frac{\Delta E}{T} \sqrt{(y^2 - 1)^2 + 1}\right) - 1} \\ &= p_0^3 g\left(\frac{\Delta E}{T}\right) \end{aligned} \quad (6.32)$$

If one expands the exponential around $y = 1$ (roton contribution) and takes for the denominator its maximum value in the interval one gets the following estimate for $(\rho_s)_{\text{roton}}$

$$\begin{aligned} (\rho_s)_{\text{roton}} &\cong \frac{p_0^3}{3\pi^2} e^{-\frac{\Delta E}{T}} \int_0^2 y^2 (y^2 - 1) dy \\ &\cong \frac{(2m\Delta E)^{3/2}}{\pi^2} e^{-\frac{\Delta E}{T}} \end{aligned} \quad (6.33)$$

Landau's estimates for $(\rho_s)_{\text{roton}}$ is⁽¹⁰⁾

$$\left(\frac{N_r}{V}\right)_{\text{Landau}} = \frac{1}{2(\pi)^{3/2}} (2m\Delta E)(mT)^{1/2} e^{-\frac{\Delta E}{T}} \quad (6.34)$$

From (6.33) and (6.34) it follows that

$$\frac{(\rho_s)_{\text{roton}}^{\text{ours}}}{\left(\frac{N_s}{V}\right)_{\text{Landau}}} = 2 \sqrt{\frac{2 \Delta E}{\pi T}} \quad (6.35)$$

Expression (6.35) shows that our definition ρ_s is essentially identical to Landau's.

The agreement between our expression is even better when compared with Landau's $(\rho^{(n)})_s$ defined in (6.15). In this case one gets, from (6.33) and (6.15)

$$\frac{(m\rho_s)_{\text{roton}}^{\text{ours}}}{(\rho^{(n)})_{\text{roton}}} = \sqrt{\frac{8}{\pi} \left(\frac{T}{\Delta E}\right)} \quad (6.36)$$

Once again the agreement with Landau's estimates is very good in the temperature interval $1.1^\circ\text{K} < T < 2, 1^\circ\text{K}$.

As a final remark in this section we would like to point out that equation (6.18) leads to the following equation for the occupation of the $\vec{p} = 0$ state:

$$\rho_c(T) = \rho_0 - \frac{2}{(2\pi)^3} \int d^3\vec{p} \frac{d\omega(\vec{p}, \rho_0)}{d\left(\frac{\vec{p}^2}{2m}\right)} N(\vec{p}) \quad (6.37)$$

where ω is given by (5.39) and $N(\vec{p})$ is defined in (6.10).

VI.4. Critical Velocity

The main experimental evidence for our interpretation of the normal component of the fluid as being associated with excitations of the ideal gas type lies in the study of the interaction of the superfluid component with the walls. As pointed out earlier by Guinzburg⁽³⁰⁾, although these excitations might not be very numerous, thus making a negligible contribution to the thermodynamic functions, they would be important in dissipating the flow. Let us see that for films of very narrow slits ($d < 10^{-4}$ cm) one can make a simple prediction for the critical velocity.

The dispersion of the momenta is given by the uncertainty principle as

$$\Delta p d \sim 1 \quad (6.38)$$

where d is the characteristic dimension of the film or slit. Following Guinzburg⁽³⁰⁾ one then has to minimize, for determining v_c , the expression

$$\left(\frac{\vec{p}^2 + \Delta p^2}{2mp}\right)_{\text{min}} \quad (6.39)$$

From (6.38) and (6.39) one gets

$$v_c d < \frac{1}{m} (1,6 \cdot 10^{-4} \text{ cm}^2/\text{seg}) \quad (6.40)$$

The bound (6.40) was a much arched one some years ago (see discussion in the Atkins book⁽⁷⁾). This bound has been obtained also by Fetter⁽³¹⁾ and Anderson⁽²⁰⁾. Although a dependence of the form $v_c d^{1/4} \approx 1$ is more appropriate from the phenomenological point of view⁽⁹⁾, some experiments exhibits the behavior predicted by (6.40)⁽⁹⁾. Result (6.40) is in a agreement with some experimental results. The conclusion is then that the normal excitations are responsible for dissipating the flow if a certain velocity is exceeded. Furthermore it explains why the critical velocity is orders of magnitude smaller than that predicted by the phonon or roton spectrum.

VII. THE λ -POINT

As pointed out in Chapter III one of the basic ingredients of our approach to Bose-Einstein condensation, equation (5.4), is the determination of the occupation of the macroscopic state with $\vec{p}=0$. The critical temperature is the one for which, from equation (4.21), this occupation is zero. By using this criterion, one gets the following equation for determining the critical temperature

$$T_c \frac{\partial}{\partial \rho_0} \left[\int \frac{d^3 \vec{p}}{(2\pi)^3} \ln \left(1 - e^{-\frac{\omega(\vec{p}, \rho_0)}{T_c}} \right) \right]_{\rho_0=0} = \Delta E \quad (7.1)$$

The critical temperature equation (7.1) can be written under a fairly simple form if one uses (6.10) and (5.10):

$$2 \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d\varepsilon(\vec{p}, \rho_0=0)}{d\left(\frac{\vec{p}^2}{2m}\right)} N(\vec{p}) = \rho_{\text{He}} \quad (7.2)$$

where $N(\vec{p})$ is defined in (6.10).

Clearly equation (7.2) can also be obtained from (6.34) by imposing $\rho_c(T_c) = 0$.

By using the dispersion relation in (5.39) in (7.1) or, equivalently, (7.2) one gets the following expression for T_c

$$\int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{\left(\frac{\vec{p}^2}{2m} - \Delta E\right)}{\sqrt{\left(\frac{\vec{p}^2}{2m} - \Delta E\right)^2 + \Delta E^2} \left[\exp \beta_c \sqrt{\left(\frac{\vec{p}^2}{2m} - \Delta E\right)^2 + (\Delta E)^2} - 1 \right]} = \frac{m\Delta E}{16\pi a} \quad (7.3)$$

After some changes of variables expression (7.3) can be written as

$$f\left(\frac{\Delta E}{T_c}\right) = \frac{\pi}{32\sqrt{2} a \sqrt{m\Delta E}} \quad (7.4)$$

where

$$f\left(\frac{\Delta E}{T_c}\right) = \int_0^\infty \frac{dy y^2 (y^2 - 1)}{\sqrt{(y^2 - 1)^2 + 1}} \frac{1}{\left\{ \exp \left[\frac{\Delta E}{T_c} \sqrt{(y^2 - 1)^2 + 1} \right] - 1 \right\}} \quad (7.5)$$

A numerical solution of equation (7.4) leads us to

$$T_c = 4,5^\circ\text{K} \quad (7.6)$$

Although result (7.6) is a factor 2 of the experimental result it shows clearly that Bose-Einstein condensation is indeed related to the λ -point. If one is able to understand what is happening at the phase transition one would get a better result for the critical temperature.

We have shown that at zero temperature the particles in the condensate are binned. Let us analyze under which conditions there would be unbinding of particles as result of temperature increase. This would happen for T_c given by

$$\frac{3}{2} T_c = \frac{\Delta}{2} \quad (7.7)$$

Condition (7.7) implies

$$T_c = 3,3^\circ\text{K} \quad (7.8)$$

If our argument that the critical temperature is the one for which there is unbinding of the atoms in the condensate, then a better estimate of the critical temperature would be

$$N \frac{\Delta}{2} = V \int d^3 \vec{p} \frac{\vec{p}^2}{2m} N(\vec{p}) \quad (7.9)$$

that is, when the average kinetic energy is equal to the total energy of the bound system

$$\rho_{\text{He}} \left(\frac{\Delta}{2}\right) = \int d^3 \vec{p} \frac{\vec{p}^2}{2m} N(\vec{p}) \quad (7.10)$$

If one takes into account only the roton contribution to (7.10) and writes

$$\rho_{\text{He}}\left(\frac{\Delta}{2}\right) \cong \int d^3\vec{p} \frac{\vec{p}^2}{2m} N_r(\vec{p}) \quad (7.11)$$

with N_r given by (6.12). Expression (7.11) is just a variant of Landau's condition $(\rho_n)_r = \rho_{\text{He}}$ which implies

$$\rho_{\text{He}}\left(\frac{3T}{2}\right) = \int d^3\vec{p} \frac{\vec{p}^2}{2m} N_r \quad (7.12)$$

The idea that at the critical temperature there is an unbinding of the N -body bound state leads us to the determination of the critical temperature if one imposes further that at the critical temperature

$$\Delta E(T_c) = 0 \quad (7.13)$$

The requirement (7.13) is equivalent to saying that the binding energy at the critical temperature is zero. Formally condition (7.13) is equivalent, within the field theoretical context, to the condition

$$\rho_c(T_c) = 0 \quad (7.14)$$

since condition (7.13) only requires that the linear coefficient of the expansion of the free energy as a function of the condensate vanishes. We should then analyse better this question by using a field theoretical approach. For the moment we just note that if one imposes that the gap is zero then equation (7.2) can be written as

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\exp\left(\frac{\vec{p}^2}{2m T_c}\right) - 1} = \frac{\rho_{\text{He}}}{2} \equiv \rho_{\text{gas}}(T_c) \quad (7.15)$$

The striking feature of equation (7.15) is its similarity to the critical temperature equation derived by London within the free Bose gas approximation. The critical temperature determined by London T_c^L was

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\exp\left(\frac{\vec{p}^2}{2m T_c^L}\right) - 1} = \rho_{\text{He}} \equiv \rho_{\text{gas}}(T_c) \quad (7.16)$$

From (7.16) and (7.15) it follows that

$$\frac{T_c^{\text{ours}}}{T_c^{\text{London}}} = 2^{-2/3} \quad (7.17)$$

From (7.17) and (2.12) it follows

$$T_c = 2.0^\circ\text{K} \quad (7.18)$$

Clearly result (7.15) could be inferred directly from (7.2) by imposing that when there is no Bose-Einstein condensation the dispersion relation is the one of free particles. That is, if one assumes that for (7.2) only normal particles contribute we get (7.15).

Finally we can see that for the critical temperature determined from condition (7.13) it follows from (6.18) that the critical temperature is the one for which

$$\rho_n(T_c) = \rho_{\text{He}} \quad (7.19)$$

Result (7.19) is verified experimentally, and it follows from (6.24) and (7.15).

We leave to a future publication a better description of the λ -point (computation of the specific heat, viscosity coefficient and the heat conductivity) as well as a derivation using the field theoretical approach of the gap equation.

VIII. CONCLUSIONS

In this paper we have studied the imperfect Bose gas at very low temperatures. We have shown that the relevant parameters for understanding the thermodynamical properties of such a system is the scattering length and the depth of the potential. Our effective Hamiltonian to He^4 is built up from these two parameters which are taken directly from the two-body potential associated to the interaction of Helium atoms. Our model does not contain free parameters. It is then a microscopic theory.

We achieve, by using the effective Hamiltonian, a complete description of Helium superfluid for $T < 1.0^\circ\text{K}$. This is the range of validity of the low temperature effective Hamiltonian (3.20). For $T \gtrsim 1.0$ although we have not succeeded in writing an effective Hamiltonian in this range we have succeeded in getting the roton spectrum by arguing that the system is in the process of restoring the symmetry. This entails, formally, a coupling among the modes. By using the relativistic prescription we have been able to get the roton spectrum.

The main achievement of our paper, besides getting the relevant features of Helium at low temperatures such as spectrum, binding energy, Helium density, is the complete determination of relevant microscopic quantities of Helium superfluid in terms of three parameters.

The main results of our paper are contained in the first column of Table (1). They express the relevant physical quantity in terms of the three parameters.

Table 1

	Our Model	Predicted Value	Experimental
ρ_{He}	$\frac{m\Delta E}{8\pi a}$	$1.52 \cdot 10^8 \text{ (eV)}^3$	$1.6 \cdot 10^8 \text{ (eV)}^3$
ϵ	$-\frac{\Delta E}{2}$	$-5.0 \cdot 10^{-4} \text{ eV}$	$-5.5 \cdot 10^{-4} \text{ eV}$
v_s	$\sqrt{\frac{\Delta E}{m}}$	$5.3 \cdot 10^{-7}$	$7 \cdot 10^{-7}$
p_0	$\sqrt{2m\Delta E}$	$2.8 \cdot 10^3 \text{ eV}$	$3.8 \cdot 10^3 \text{ eV}$
Δ	ΔE	$10.2 \cdot 10^{-4} \text{ eV}$	$9 \cdot 10^{-4} \text{ eV}$
μ	$\frac{1}{2}m$	$\frac{1}{2}m$	$0.3 < \mu < 0.7m$
$P(0)$	$\frac{m\Delta E^2}{16\pi a}$	$6 \cdot 10^4 \text{ (eV)}^4$	$4.8 \cdot 10^4 \text{ (eV)}^4$
$v_c d$	$< \frac{1}{m}$	$< \frac{1}{3.5} \cdot 10^{-9} \text{ eV}^{-1}$	$< \frac{1}{3.5} \cdot 10^{-9} \text{ eV}$

For comparison we present in the second row of Table 1 the predicted values for these parameters by taking $a \cong 2 \text{ \AA}$, $\Delta E \sim 10.2 \cdot 10^{-4} \text{ eV}$ and $3.5 \cdot 10^9 \text{ eV}$. In the third row we present the experimental values.

Our conclusion is that we succeeded in getting, from a microscopic theory not only the essential features of Helium superfluid but we have also succeeded in getting a good account of a very large numbers of macroscopic parameters of Helium superfluid at low temperatures.

Other result that is worth emphasizing is the identification of the normal component of the fluid as Helium atoms whose momentum is larger than $\sqrt{2m\Delta E}$. This confirms a conjecture first made by Tisza⁽¹³⁾ on the nature of the normal fluid (this is his postulate (c)). Such a finding allows us to write explicit expressions for all thermodynamical variables involving the normal component. In particular one can write

$$\frac{\rho_n(T)}{\rho_n(T_c)} = \left(\frac{T}{T_c}\right)^{3/2} \frac{h\left(\frac{\Delta E}{T}\right)}{h\left(\frac{\Delta E}{T_c}\right)} \quad (7.20)$$

The dependence of $\rho_n(T)/\rho_n(T_c)$ exhibited in fig (2) is in an excellent agreement

with Andronikashvili determination of $\rho_n(T)/\rho$.

Finally we comment on the determination of the critical temperature. In our first estimate we have taken the gap as independent of temperature. This is obviously true in the "classical approximation". That is, one expects that as a result of quantum thermal fluctuations the gap disappears. Within the constant gap approximation we get $T_c = 4.5^\circ\text{K}$. However if one imposes in some of our equations that at the critical temperature

$$\begin{aligned}\rho_c(T_c) &= 0 \\ \Delta E(T_c) &= 0\end{aligned}$$

Then, from (7.15) it follows that

$$T_c = \frac{2\pi}{m} \left[\zeta^{-1}(3/2) \frac{\rho_{\text{He}}}{2} \right]^{2/3} \cong 2^\circ\text{K}$$

This is a very good estimate for the λ -point (ζ above is the Riemann zeta function). From (6.22) and (6.18) we get

$$\rho_n(T_c) = \rho_{\text{He}}$$

Furthermore, since (6.24) predicts that

$$\frac{\rho_n}{\rho}(T \sim 1.1) \cong 0$$

one gets results that are totally in agreement with the experiments. Furthermore the above results are just Tisza's postulates (c) and (d).

In terms of the three fundamental parameters a , ΔE and m one can write for the critical temperature

$$T_c = \frac{2\pi}{m} \left(\frac{m\Delta E}{16\pi a} \frac{1}{\zeta(3/2)} \right)^{2/3}$$

whereas for the normal component

$$\frac{\rho_n(T)}{\rho_{\text{He}}} = \left(\frac{T}{T_\lambda} \right)^{3/2} \frac{\int_{\sqrt{\frac{\Delta E}{T}}} dy y^2 \frac{1}{e^{y^2} - 1}}{\int_{\sqrt{4.13}}^{\infty} dy y^2 \frac{1}{e^{y^2} - 1}}$$

It will be then important to analyse the gap equation taking into account quantum corrections. The simple expression presented by us shows that it will be possible to reproduce all phenomenologically relevant features of Helium superfluid close to the λ -point. These questions as well as the determination of transport coefficients is currently pursued by us.

One of the byproducts of our paper was to shed some light on the Landau-Tisza controversy⁽¹⁴⁾. In fact, from (6.18) it follows that Helium density is composed of a uniform background (Bogoliubov's condensate) plus excitations. However, some excitations are just Helium atoms. We have shown that the normal component is associated to these atoms. With regard to the separation of the Helium density Landau was correct. However, as far the identification of the normal component is concerned we have checked that Tisza's view is more appropriate.

Our approach to Bose-Einstein condensation makes a simple prediction for the equation of state at zero temperature

$$\frac{P}{\rho} = -\varepsilon$$

which deserves to be tested experimentally.

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APPENDIX A

Aimed at the justification of the effective Hamiltonian (3.7) let us separate ρ in (3.3) into two terms and write, formally

$$\rho = \rho_s(\vec{x}) + \rho_n(\vec{x}) \quad (\text{A.1})$$

After substituting (A.1) into (3.2) we get for the Hamiltonian density,

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \rho_s(\vec{x}) V(\vec{x} - \vec{x}') \rho_s(\vec{x}') + \rho_s(\vec{x}) V(\vec{x} - \vec{x}') \rho_n(\vec{x}) + \frac{1}{2} \rho_n(\vec{x}) V(\vec{x} - \vec{x}') \rho_n(\vec{x}') \\ &\equiv \mathcal{H}_{ss} + \mathcal{H}_{sn} + \mathcal{H}_{nn} \end{aligned} \quad (\text{A.2})$$

Let us associate ρ_s to an uniform background (condensate) and write

$$\rho_s(\vec{x}) = \rho_0 \quad (\text{A.3})$$

For an uniform density one can write for H_{ss} defined in (A.2)

$$H_{ss} \equiv \frac{1}{2} \int \rho_s(\vec{x}) V(\vec{x} - \vec{x}') \rho_s(\vec{x}') \equiv \frac{4\pi a}{m} V \rho_0^2 \quad (\text{A.4})$$

Expression (A.4) implies that particles in the uniform condensate interacts among themselves as hard spheres and that the only relevant parameter describing their interaction is the scattering length.

Let us analyse now the interaction of particles in the condensate with those not belonging to the condensate. That is, we will be interested in studying H_{sn} defined by

$$H_{sn} = \iint \rho_s(\vec{x}) V(\vec{x} - \vec{x}') \rho_n(\vec{x}) d^3\vec{x} d^3\vec{x}' \quad (\text{A.5})$$

In order to introduce a sensible approximation to (A.5) we just recall that in the low temperature limit a single particle is insensitive to some details of the potential. In fact, within the semiclassical approximation the only relevant region of the potential is the point in which the potential has a minimum.

If we expands V around the minimum one gets

$$H_{sn} = V_{\min} \int d^3\vec{x} \rho_s(\vec{x}) \int d^3\vec{x}' \rho_n(\vec{x}') + \text{higher momentum distribution terms} \quad (\text{A.6})$$

For an uniform condensate and if we neglect the higher momentum particle distribution terms (that is, neglecting details of the potential) one gets an approximate expression for H_{sn} :

$$H_{sn} = V \rho_0 \left(V_{\min} \int d^3\vec{x} \rho_n(\vec{x}) \right) \quad (\text{A.7})$$

The interaction of particles in the condensate with a single particle not belonging to the condensate ($H_{sn}^{(1)}$) will be then, from (A.7):

$$H_{sn}^{(1)} = V_{\min} V \rho_0 \quad (\text{A.8})$$

If one adds (A.4) with (A.8) one would get a simple expression for the interaction of particles in the condensate, that is

$$H_s \equiv H_{ss} + H_{sn}^{(1)} = V \left[\left(\frac{4\pi a}{m} \right) \rho_0^2 + V_{\min} \rho_0 \right] \quad (\text{A.9})$$

Expression (A.9) for H_s is equivalent to (3.7) with λ given by (3.6) and ΔE given by (3.8).

Since V_{\min} is negative, H_s satisfy the requirement of being a quantum liquid since as function of the condensate one has a minimum for

$$\rho_0 = - \frac{V_{\min}}{8\pi a} m \quad (\text{A.10})$$

The identification of ρ_0 with ρ_{He} leads to

$$\rho_{\text{He}} \cong 1.52 \cdot 10^8 (\text{eV})^3 \quad (\text{A.11})$$

in agreement with (2.7).

The conclusion is that our effective Hamiltonian for the description of Helium-4 incorporates one global aspect of the potential (the scattering length) and a local aspect of the potential (minimum of the potential).

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FIGURE CAPTIONS

Fig. 1 - Potential energy between two He-4 atoms as function of distance. The relevant parameters, ΔE and a , are shown explicitly here.

Fig. 2 - ρ_n/ρ as function of temperature. Open circles are experimental values of Andronikashvili. Dark circles were obtained from $\rho_n(T)\rho(T = 2.2^\circ\text{K})$ as proposed in this paper. Solid line represents the fit $\rho_n/\rho = (T/T_\lambda)^{5.6}$.

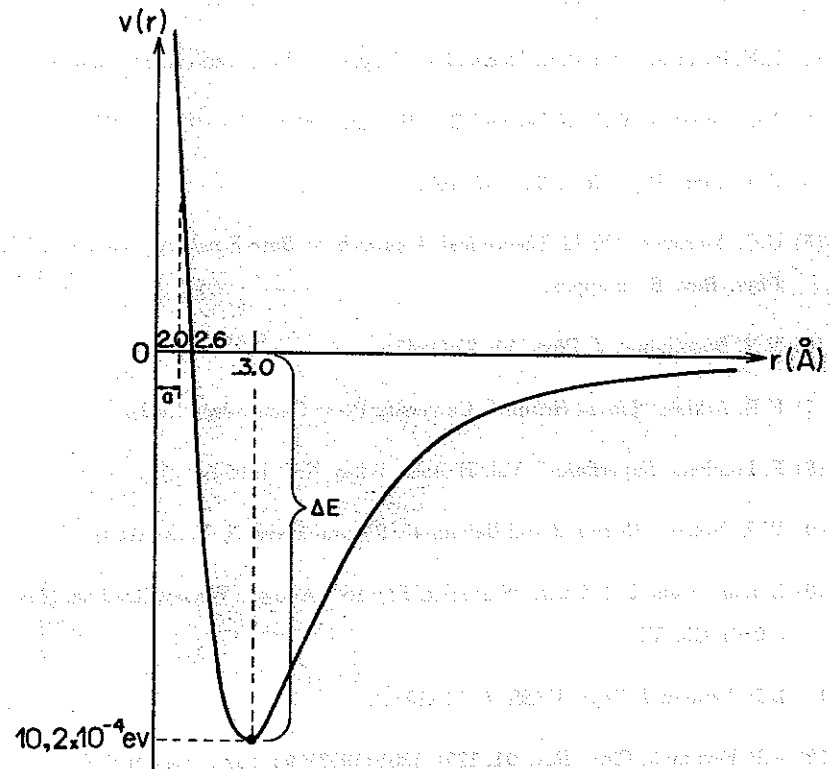


Fig. 1

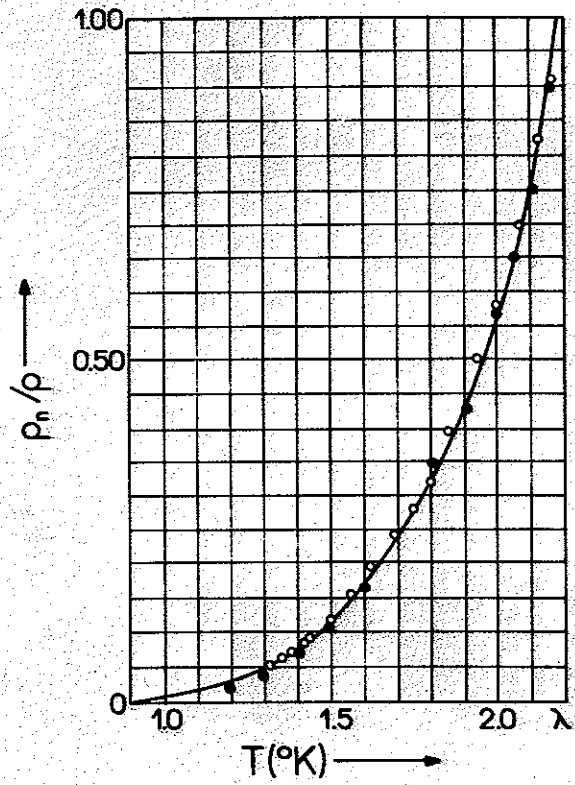


Fig. 2