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Current Algebra of WZNW Models at and away from Criticality

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Abstract

We derive the current algebra of principal chiral models with a Wess-Zumino term. At the critical coupling where the model becomes conformally invariant (Wess-Zumino-Novikov-Witten theory), this algebra reduces to two commuting Kac-Moody algebras, while in the limit where the coupling constant is taken to zero (ordinary chiral model), we recover the current algebra of that model. In this way, the latter is explicitly realized as a deformation of the former, with the coupling constant as the deformation parameter.

Algebraic methods have during the last few years been used extensively to unravel the dynamical structure of two-dimensional field theoretical models, especially integrable models and conformal field theories. In the case of conformally invariant models, a purely algebraic approach (using, e.g., the representation theory of the Virasoro algebra and of Kac-Moody algebras, together with that of braid groups) may even lead to a complete solution of the theory.

In the present paper, we shall investigate the current algebra for the principal chiral model with a Wess-Zumino term. This model [1] contains a free coupling constant λ and, for special values of λ, contains the conformally invariant WZNW model as well as the ordinary chiral model. Therefore, the current algebra derived below is a common generalization of the current algebras for these two special cases. Now for the WZNW model, the current algebra is well-known to consist of two commuting Kac-Moody algebras, while for the ordinary chiral model, it is a new kind of algebra which – although having been known in part for some time (cf., e.g., [2, pp. 323/324]) – has only recently been specified completely [3] and whose mathematical structure is still far from being understood. The hope is that through the "interpolating" current algebra derived here, the well-developed theory of Kac-Moody algebras may shed some light on the current algebra for integrable chiral models in general: this has been the main motivation behind the present work.

We begin by fixing our conventions. The target space for the chiral models to be considered in this paper will be a simple Lie group G (which is usually, though not necessarily, assumed to be compact) with Lie algebra g, and we shall use the trace tr in some irreducible representation to define a) the invariant scalar product $(.,.)$ on g, normalized so that the long roots have length $\sqrt{2}$, and b) the invariant closed three-form ω on g giving rise to the Wess-Zumino term. Explicitly, for $X, Y, Z \in g$,

$$(X, Y) = -\operatorname{tr}(XY) \ , \tag{1}$$

while

$$\omega(X, Y, Z) = \frac{1}{4\pi} \operatorname{tr}(X[Y, Z]) \ . \tag{2}$$

Obviously, $(.,.)$ and ω extend to a biinvariant metric $(.,.)$ on G and to a biinvariant three-form ω on G, respectively: the latter can alternatively be represented in terms of the left invariant Maurer-Cartan form $g^{-1}dg$ or right invariant Maurer-Cartan form $dg\,g^{-1}$ on G, as follows:

$$\omega = \frac{1}{12\pi} \operatorname{tr}(g^{-1}dg)^3 = \frac{1}{12\pi} \operatorname{tr}(dg\,g^{-1})^3 \ . \tag{3}$$

(Due to the Maurer-Cartan structure equation, this representation implies that ω is indeed a closed three-form on G, and the normalization in eqns (2) and (3) is chosen so that $\omega/2\pi$ generates the third de Rham cohomology group $H^3(G, \mathbb{Z})$ of G over the integers, at least when G is simply connected; cf. ref. [4, p. 129 and Appendix 1].

Finally, the minus sign in eqn (1) is introduced to ensure positive definiteness when G is compact.)

In most of what follows, we shall work in terms of (arbitrary) local coordinates u^i on G, representing the metric $(.,.)$ by its components g_{ij} and the three-form ω by its components ω_{ijk}. Then the total action is the sum

$$S = S_{CH} + n S_{WZ} \qquad (4)$$

of the action for the ordinary chiral model

$$S_{CH} = \frac{1}{2\lambda^2} \int_\Sigma d^2x \, \eta^{\mu\nu} \, g_{ij}(\varphi) \, \partial_\mu\varphi^i \, \partial_\nu\varphi^j \quad , \qquad (5)$$

with coupling constant λ, and the Wess-Zumino term

$$S_{WZ} = \frac{1}{6} \int_B d^3x \, \epsilon^{\kappa\lambda\mu} \, \omega_{ijk}(\tilde\varphi) \, \partial_\kappa\tilde\varphi^i \, \partial_\lambda\tilde\varphi^j \, \partial_\mu\tilde\varphi^k = \int_B \tilde\varphi^\star\omega \quad . \qquad (6)$$

Here, φ and $\tilde\varphi$ are the basic field and the extended field of the theory, respectively, i.e., φ is a (smooth) map from a fixed two-dimensional Lorentz manifold Σ to G and $\tilde\varphi$ is a (smooth) map from an appropriate three-dimensional manifold B to G, chosen so that Σ is the boundary of B and φ is the restriction of $\tilde\varphi$ to that boundary:

$$\Sigma = \partial B \quad , \quad \varphi = \tilde\varphi|_\Sigma \quad .$$

The conformally invariant WZNW model is obtained at $\lambda = \sqrt{4\pi/|n|}$, while the ordinary chiral model can be recovered in the limit $\lambda \to 0$. Note that if ω were exact, we could write $\omega = d\alpha$ to obtain

$$S_{WZ} = \frac{1}{2} \int_\Sigma d^2x \, \epsilon^{\mu\nu} \, \alpha_{ij}(\varphi) \, \partial_\mu\varphi^i \, \partial_\nu\varphi^j = \int_\Sigma \varphi^\star\alpha \quad . \qquad (7)$$

But of course this is not possible globally, i.e., the α_{ij} appearing in this formula are neither unique nor can they be chosen so as to become the components of a globally well-defined two-form on G with respect to the u^i. Still, calculations involving quantities that arise from local variations of the action can be performed as if this were the case, and may lead to results that do not depend on any artificial choices. For example, recall that in the ordinary chiral model, the canonically conjugate momenta π_i derived from the action S_{CH} are simply given by

$$\pi_i = \frac{1}{\lambda^2} \, g_{ij}(\varphi) \, \dot\varphi^j \qquad (8)$$

and satisfy the canonical commutation relations

$$\{\varphi^i(x), \varphi^j(y)\} = 0 \quad , \quad \{\pi_i(x), \pi_j(y)\} = 0 \quad ,$$
$$\{\varphi^i(x), \pi_j(y)\} = \delta^i_j \delta(x-y) \quad . \qquad (9)$$

Similarly, in the chiral model with a Wess-Zumino term, written in the form (7), the canonically conjugate momenta $\hat\pi_i$ derived from the action S are given by

$$\hat\pi_i = \pi_i + n \, \alpha_{ij}(\varphi) \, \varphi'^j \qquad (10)$$

and satisfy the canonical commutation relations

$$\{\varphi^i(x), \varphi^j(y)\} = 0 \quad , \quad \{\hat\pi_i(x), \hat\pi_j(y)\} = 0 \quad ,$$
$$\{\varphi^i(x), \hat\pi_j(y)\} = \delta^i_j \delta(x-y) \quad . \qquad (11)$$

Note, however, that in contrast to the π_i, the $\hat\pi_i$ do not behave naturally under local coordinate transformations on G, so that the canonical commutation relations (11) between the φ^i and the $\hat\pi_j$ look non-covariant. This suggests to consider instead the commutation relations between the φ^i and the π_j, which are covariant, but exhibit non-vanishing Poisson brackets between the π_i. Indeed, it follows from (11) that

$$\{\pi_i(x), \pi_j(y)\} = \{\hat\pi_i(x) - n\,\alpha_{ik}(\varphi(x))\,\varphi'^k(x) \, , \, \hat\pi_j(y) - n\,\alpha_{jl}(\varphi(y))\,\varphi'^l(y)\}$$
$$= n \left(-\{\hat\pi_i(x), \, \alpha_{jl}(\varphi(y))\,\varphi'^l(y)\} \right.$$
$$\left. - \{\alpha_{ik}(\varphi(x))\,\varphi'^k(x), \, \hat\pi_j(y)\} \right)$$
$$= n \left(+ \partial_i\alpha_{jl}(\varphi(x))\,\varphi'^l(x)\,\delta(x-y) - \alpha_{ji}(\varphi(y))\,\delta'(x-y) \right.$$
$$\left. - \partial_j\alpha_{ik}(\varphi(x))\,\varphi'^k(x)\,\delta(x-y) - \alpha_{ij}(\varphi(x))\,\delta'(x-y) \right)$$
$$= + n \left(\partial_i\alpha_{jk} + \partial_j\alpha_{ki} \right)(\varphi(x))\,\varphi'^k(x)\,\delta(x-y)$$
$$+ n \, \partial_k\alpha_{ij}(\varphi(x))\,\varphi'^k(x)\,\delta(x-y) \quad ,$$

so in the presence of the Wess-Zumino term, the commutation relations between the φ^i and the π_j read

$$\{\varphi^i(x), \varphi^j(y)\} = 0 \quad , \quad \{\varphi^i(x), \pi_j(y)\} = \delta^i_j \delta(x-y) \quad ,$$
$$\{\pi_i(x), \pi_j(y)\} = n\,\omega_{ijk}(\varphi(x))\,\varphi'^k(x)\,\delta(x-y) \quad . \qquad (12)$$

They are obviously covariant (all expressions behave naturally under local coordinate transformations on G), since ω is a globally well-defined three-form on G.

To derive the desired current algebra, we recall next that the model under consideration has an obvious global invariance under the product group $G_L \times G_R$, which acts on G according to

$$g \longrightarrow (g_L, g_R) \cdot g = g_L \, g \, g_R^{-1} \quad . \qquad (13)$$

This action of the Lie group $G_L \times G_R$ induces a representation of the corresponding Lie algebra $g_L \oplus g_R$ by vector fields, associating with each generator $X = (X_L, X_R)$ in $g_L \oplus g_R$ the fundamental vector field X_G on G given by

$$X_G(g) = X_L g - g X_R \quad \text{for } g \in G \quad . \qquad (14)$$

As usual, invariance of the action leads to conserved Noether currents taking values in $g_L \oplus g_R$ and denoted by j_μ for the ordinary chiral model and by $\hat{j}_\mu$ for the chiral model with a Wess-Zumino term. Explicitly, we have, for $X = (X_L, X_R)$ in $g_L \oplus g_R$,

$$(j_\mu, X) = -\frac{1}{\lambda^2} g_{ij}(\varphi) \partial_\mu \varphi^i X_G^j(\varphi) \quad , \tag{15}$$

while

$$(\hat{j}_\mu, X) = -\left(\frac{1}{\lambda^2} g_{ij}(\varphi) \partial_\mu \varphi^i + n \alpha_{ij}(\varphi) \epsilon_{\mu\nu} \partial^\nu \varphi^i\right) X_G^j(\varphi) \quad . \tag{16}$$

In addition, an important role is played by the scalar field j introduced in ref. [3], defined by

$$(j, X \otimes Y) = \frac{1}{\lambda^2} g_{ij}(\varphi) X_G^i(\varphi) Y_G^j(\varphi) \quad . \tag{17}$$

(Note the additional factors $\frac{1}{\lambda^2}$, which were absent in ref. [3].)

The commutation relations of the Noether currents j_μ and $\hat{j}_\mu$ under Poisson brackets can now be computed directly. Note again, however, that in contrast to the j_μ, the $\hat{j}_\mu$ do not behave naturally under local coordinate transformations on G, so that their commutation relations look non-covariant. This suggests to replace them by appropriate covariant currents J_μ which, as it turns out, can be written entirely in terms of the Noether currents j_μ for the ordinary chiral model (the exact definition will be given below): it is the commutation relations of these covariant currents J_μ that form the current algebra we wish to compute (or at least an important part thereof). The most efficient way of arriving at the desired result is therefore to calculate, as an intermediate step, the commutation relations of the Noether currents j_μ, using the commutation relations (12): this can be done along the lines of ref. [3]. In fact, for the ordinary chiral model the calculation proceeds in exactly the same manner as outlined there, while for the chiral model with a Wess-Zumino term, the only commutation relation that changes is the Poisson bracket of two j_0's, which picks up an additional term due to the fact that the π_i no longer commute; thus:

$$\begin{aligned}
\{(j_0(x), X), (j_0(y), Y)\} &= \{\pi_i(x) X_G^i(\varphi(x)), \pi_j(y) Y_G^j(\varphi(y))\} \\
&= \pi_i(x)\left(\partial_j X_G^i(\varphi(x)) Y_G^j(\varphi(x)) - X_G^j(\varphi(x)) \partial_j Y_G^i(\varphi(x))\right) \delta(x-y) \\
&\quad + n \omega_{ijk}(\varphi(x)) \varphi'^k(x) X_G^i(\varphi(x)) Y_G^j(\varphi(x)) \delta(x-y) \\
&= \pi_i(x) [X,Y]_G^i(\varphi(x)) \delta(x-y) \\
&\quad + n \omega(\varphi(x))(\varphi'(x), X_G(\varphi(x)), Y_G(\varphi(x))) \delta(x-y) \\
&= -(j_0(x), [X,Y]) \delta(x-y) \\
&\quad + n \omega(\varphi(x))(\varphi'(x), X_L\varphi(x) - \varphi(x)X_R, Y_L\varphi(x) - \varphi(x)Y_R) \delta(x-y)
\end{aligned} \tag{18}$$

For later reference, we also list the commutation relations which have remained unchanged:

$$\begin{aligned}
\{(j_0(x), X),\ (j_1(y), Y)\} \\
= -(j_1(x), [X,Y]) \delta(x-y) + (j(y), X \otimes Y) \delta'(x-y) \quad , \tag{19}
\end{aligned}$$

$$\{(j_1(x), X),\ (j_1(y), Y)\} = 0 \quad , \tag{20}$$

$$\begin{aligned}
\{(j_0(x), X),\ (j(y), Y \otimes Z)\} \\
= -(j(x), [X,Y] \otimes Z + Y \otimes [X,Z]) \delta(x-y) \quad , \tag{21}
\end{aligned}$$

$$\{(j_1(x), X),\ (j(y), Y \otimes Z)\} = 0 \quad . \tag{22}$$

(The additional factors $\frac{1}{\lambda^2}$ drop out completely: they have been absorbed into the normalizations of the j_μ and j.)

Before proceeding further, we find it convenient to pass to more standard notation, writing g and $\tilde{g}$, rather than φ and $\tilde{\varphi}$, for the basic field and the extended field of the theory, respectively, and using the explicit definitions (1) of the metric $(.,.)$ on G and (2) of the three-form ω on G. Then

$$S_{CH} = -\frac{1}{2\lambda^2} \int d^2x\ \eta^{\mu\nu} \operatorname{tr}\left(g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\right) \quad , \tag{23}$$

while

$$S_{WZ} = \frac{1}{4\pi} \int_0^1 dr \int d^2x\ \epsilon^{\mu\nu} \operatorname{tr}\left(\tilde{g}^{-1}\partial_r\tilde{g}\,\tilde{g}^{-1}\partial_\mu\tilde{g}\,\tilde{g}^{-1}\partial_\nu\tilde{g}\right) \quad . \tag{24}$$

(Here, the extended field $\tilde{g}$ is assumed to be constant outside a tubular neighborhood $\Sigma \times [0,1]$ of the boundary Σ of B, and r is the coordinate normal to the boundary.) Next, we decompose the currents j_μ and J_μ, both of which take values in $g_L \oplus g_R$, into left and right currents, all of which take values in g: $j_\mu = (j_\mu^L, j_\mu^R)$, $J_\mu = (J_\mu^L, J_\mu^R)$. Explicitly,

$$j_\mu^L = -\frac{1}{\lambda^2} \partial_\mu g\, g^{-1} \quad , \tag{25}$$

$$j_\mu^R = +\frac{1}{\lambda^2} g^{-1}\partial_\mu g \quad , \tag{26}$$

and, by definition,

$$J_\mu^L = \left(\eta_{\mu\nu} + \frac{n\lambda^2}{4\pi}\epsilon_{\mu\nu}\right) j^{L\nu} = -\frac{1}{\lambda^2}\left(\eta_{\mu\nu} + \frac{n\lambda^2}{4\pi}\epsilon_{\mu\nu}\right) \partial^\nu g\, g^{-1} \quad , \tag{27}$$

$$J_\mu^R = \left(\eta_{\mu\nu} - \frac{n\lambda^2}{4\pi}\epsilon_{\mu\nu}\right) j^{R\nu} = +\frac{1}{\lambda^2}\left(\eta_{\mu\nu} - \frac{n\lambda^2}{4\pi}\epsilon_{\mu\nu}\right) g^{-1}\partial^\nu g \quad , \tag{28}$$

while the scalar field j, when viewed as taking values in the space of endomorphisms of $g_L \oplus g_R$, is given by the (2×2)-block matrix

$$j = \frac{1}{\lambda^2} \begin{pmatrix} 1 & -\text{Ad}(g) \\ -\text{Ad}(g)^{-1} & 1 \end{pmatrix} \quad . \tag{29}$$

In other words, for $X = (X_L, X_R)$ in $g_L \oplus g_R$,

$$j(X) = \frac{1}{\lambda^2} \Big(X_L - \text{Ad}(g) X_R , \ X_R - \text{Ad}(g)^{-1} X_L \Big) \quad . \tag{30}$$

It can be shown that the covariant currents J_μ defined by eqns (27,28) differ from the Noether currents $\jmath_\mu$ for the chiral model with a Wess-Zumino term by a total curl, and that current conservation (which for both types of currents has the same physical content, because a total curl is automatically conserved) is identical with the equations of motion of the theory; cf., e.g., ref. [5] for more details[1]. (Note also that the additional factors of $\frac{1}{2}$ in eqns (24) and (25) of ref. [3], as compared to eqns (25,26) and (29) above, are due to the identification, performed there, of G with the Riemannian symmetric space $G \times G / \Delta G$, which results in G being equipped with a biinvariant metric that is twice the one used here.)

Now in terms of an arbitrary basis (T_a) of g, with structure constants f_{ab}^c defined by $[T_a, T_b] = f_{ab}^c T_c$, the various currents are represented by their components

$$j_{\mu,a}^L = (j_\mu, T_a^L) = -\text{tr}(j_\mu^L T_a) \quad , \tag{31}$$

$$j_{\mu,a}^R = (j_\mu, T_a^R) = -\text{tr}(j_\mu^R T_a) \quad , \tag{32}$$

$$J_{\mu,a}^L = (J_\mu, T_a^L) = -\text{tr}(J_\mu^L T_a) \quad , \tag{33}$$

$$J_{\mu,a}^R = (J_\mu, T_a^R) = -\text{tr}(J_\mu^R T_a) \quad , \tag{34}$$

and the scalar field j by its components

$$\eta_{ab} = (j, T_a^L \otimes T_b^L) = (j, T_a^R \otimes T_b^R) = -\frac{1}{\lambda^2} \text{tr}(T_a T_b) \quad , \tag{35}$$

$$t_{ab} = (j, T_a^L \otimes T_b^R) = \frac{1}{\lambda^2} \text{tr}\Big(g^{-1} T_a g T_b\Big) \quad , \tag{36}$$

where

$$T_a^L = (T_a, 0) \quad , \quad T_a^R = (0, T_a) \quad . \tag{37}$$

With this notation, eqns (18)–(20) can easily be seen to imply the following commutation relations for the components of the currents j_μ:

[1] Our conventions here differ from those of ref. [5] in that we write λ^2 instead of λ and use $\epsilon^{01} = +1, \epsilon_{01} = -1$: thus the formulas valid for $n > 0$ ($n < 0$) there correspond to the formulas valid for $n < 0$ ($n > 0$) here.

$$\{j_{0,a}^L(x), j_{0,b}^L(y)\} = -f_{ab}^c j_{0,c}^L(x) \delta(x-y) + \frac{n\lambda^2}{4\pi} f_{ab}^c j_{1,c}^L(x) \delta(x-y) \quad , \tag{38}$$

$$\{j_{0,a}^L(x), j_{1,b}^L(y)\} = -f_{ab}^c j_{1,c}^L(x) \delta(x-y) + \eta_{ab} \delta'(x-y) \quad , \tag{39}$$

$$\{j_{1,a}^L(x), j_{1,b}^L(y)\} = 0 \quad , \tag{40}$$

$$\{j_{0,a}^R(x), j_{0,b}^R(y)\} = -f_{ab}^c j_{0,c}^R(x) \delta(x-y) - \frac{n\lambda^2}{4\pi} f_{ab}^c j_{1,c}^R(x) \delta(x-y) \quad , \tag{41}$$

$$\{j_{0,a}^R(x), j_{1,b}^R(y)\} = -f_{ab}^c j_{1,c}^R(x) \delta(x-y) + \eta_{ab} \delta'(x-y) \quad , \tag{42}$$

$$\{j_{1,a}^R(x), j_{1,b}^R(y)\} = 0 \quad , \tag{43}$$

$$\{j_{0,a}^L(x), j_{0,b}^R(y)\} = \frac{n\lambda^2}{4\pi} t_{ab}'(x) \delta(x-y) \quad , \tag{44}$$

$$\{j_{0,a}^L(x), j_{1,b}^R(y)\} = t_{ab}(y) \delta'(x-y) \quad , \tag{45}$$

$$\{j_{0,a}^R(x), j_{1,b}^L(y)\} = t_{ba}(y) \delta'(x-y) \quad , \tag{46}$$

$$\{j_{1,a}^L(x), j_{1,b}^R(y)\} = 0 \quad . \tag{47}$$

They must be supplemented by the commutation relations between the components of the currents j_μ and those of the field t, which follow from eqns (21) and (22):

$$\{j_{0,a}^L(x), t_{bc}(y)\} = -f_{ab}^d t_{dc}(x) \delta(x-y) \quad , \tag{48}$$

$$\{j_{0,a}^R(x), t_{bc}(y)\} = -f_{ac}^d t_{bd}(x) \delta(x-y) \quad , \tag{49}$$

$$\{j_{1,a}^L(x), t_{bc}(y)\} = 0 \quad , \tag{50}$$

$$\{j_{1,a}^R(x), t_{bc}(y)\} = 0 \quad , \tag{51}$$

Finally, the components of the field t commute among themselves:

$$\{t_{ab}(x), t_{cd}(y)\} = 0 \quad . \tag{52}$$

In terms of light-cone components, which for any vector a_μ are defined by $a_\pm = a_0 \pm a_1$, these commutation relations take the form

$$\{j_{\pm,a}^L(x), j_{\pm,b}^L(y)\} = -\frac{1}{2} f_{ab}^c \Big[\Big(3 \mp \frac{n\lambda^2}{4\pi}\Big) j_{\pm,c}^L(x) - \Big(1 \mp \frac{n\lambda^2}{4\pi}\Big) j_{\mp,c}^L(x)\Big] \delta(x-y)$$
$$\pm 2 \eta_{ab} \delta'(x-y) \quad , \tag{53}$$

$$\{j_{\pm,a}^R(x), j_{\pm,b}^R(y)\} = -\frac{1}{2} f_{ab}^c \Big[\Big(3 \pm \frac{n\lambda^2}{4\pi}\Big) j_{\pm,c}^R(x) - \Big(1 \pm \frac{n\lambda^2}{4\pi}\Big) j_{\mp,c}^R(x)\Big] \delta(x-y)$$
$$\pm 2 \eta_{ab} \delta'(x-y) \quad , \tag{54}$$

$$\{j_{\pm,a}^L(x), j_{\pm,b}^R(y)\} = \pm \Big[\Big(1 \mp \frac{n\lambda^2}{4\pi}\Big) t_{ab}(x) + \Big(1 \pm \frac{n\lambda^2}{4\pi}\Big) t_{ab}(y)\Big] \delta'(x-y) \quad , \tag{55}$$

$$\{j_{\pm,a}^L(x), j_{\mp,b}^L(y)\} = -\frac{1}{2} f_{ab}^c \left[\left(1 - \frac{n\lambda^2}{4\pi} \right) j_{+,c}^L(x) + \left(1 + \frac{n\lambda^2}{4\pi} \right) j_{-,c}^L(x) \right] \delta(x-y), \quad (56)$$

$$\{j_{\pm,a}^R(x), j_{\mp,b}^R(y)\} = -\frac{1}{2} f_{ab}^c \left[\left(1 + \frac{n\lambda^2}{4\pi} \right) j_{+,c}^R(x) + \left(1 - \frac{n\lambda^2}{4\pi} \right) j_{-,c}^R(x) \right] \delta(x-y), \quad (57)$$

$$\{j_{\pm,a}^L(x), j_{\mp,b}^R(y)\} = \mp \left(1 \mp \frac{n\lambda^2}{4\pi} \right) t_{ab}'(x) \delta(x-y), \quad (58)$$

$$\{j_{\pm,a}^R(x), j_{\mp,b}^L(y)\} = \mp \left(1 \pm \frac{n\lambda^2}{4\pi} \right) t_{ba}'(x) \delta(x-y), \quad (59)$$

and

$$\{j_{\pm,a}^L(x), t_{bc}(y)\} = -f_{ab}^d t_{dc}(x) \delta(x-y), \quad (60)$$

$$\{j_{\pm,a}^R(x), t_{bc}(y)\} = -f_{ac}^d t_{bd}(x) \delta(x-y), \quad (61)$$

together with eqn (52).

Passing to the commutation relations for the currents J_μ instead of the currents j_μ now trivial, because according to eqns (27,28), their light-cone components just differ by a normalization factor:

$$J_\pm^L = \left(1 \pm \frac{n\lambda^2}{4\pi} \right) j_\pm^L, \quad (62)$$

$$J_\pm^R = \left(1 \mp \frac{n\lambda^2}{4\pi} \right) j_\pm^L, \quad (63)$$

We nevertheless write down the corresponding formulas, because they constitute the main result of this paper; they read

$$\{J_{\pm,a}^L(x), J_{\pm,b}^L(y)\}$$

$$= -\frac{1}{2} \left(1 \pm \frac{n\lambda^2}{4\pi} \right) f_{ab}^c \left[\left(3 \mp \frac{n\lambda^2}{4\pi} \right) J_{\pm,c}^L(x) - \left(1 \pm \frac{n\lambda^2}{4\pi} \right) J_{\mp,c}^L(x) \right] \delta(x-y)$$

$$\pm 2 \left(1 \pm \frac{n\lambda^2}{4\pi} \right)^2 \eta_{ab} \delta'(x-y), \quad (64)$$

$$\{J_{\pm,a}^R(x), J_{\pm,b}^R(y)\}$$

$$= -\frac{1}{2} \left(1 \mp \frac{n\lambda^2}{4\pi} \right) f_{ab}^c \left[\left(3 \pm \frac{n\lambda^2}{4\pi} \right) J_{\pm,c}^R(x) - \left(1 \mp \frac{n\lambda^2}{4\pi} \right) J_{\mp,c}^R(x) \right] \delta(x-y)$$

$$\pm 2 \left(1 \mp \frac{n\lambda^2}{4\pi} \right)^2 \eta_{ab} \delta'(x-y), \quad (65)$$

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{J^L_{\pm,a}(x), J^R_{\pm,b}(y)}

$$\{J^L_{\pm,a}(x),\, J^R_{\pm,b}(y)\}$$
$$= \pm\left(1-\left(\frac{n\lambda^2}{4\pi}\right)^2\right)\left[\left(1\mp\frac{n\lambda^2}{4\pi}\right)t_{ab}(x) + \left(1\pm\frac{n\lambda^2}{4\pi}\right)t_{ab}(y)\right]\delta'(x-y)\,, \qquad (66)$$

$$\{J^L_{\pm,a}(x),\, J^L_{\mp,b}(y)\}$$
$$= -\frac{1}{2}f^c_{ab}\left[\left(1-\frac{n\lambda^2}{4\pi}\right)^2 J^L_{+,c}(x) + \left(1+\frac{n\lambda^2}{4\pi}\right)^2 J^L_{-,c}(x)\right]\delta(x-y)\,, \qquad (67)$$

$$\{J^R_{\pm,a}(x),\, J^R_{\mp,b}(y)\}$$
$$= -\frac{1}{2}f^c_{ab}\left[\left(1+\frac{n\lambda^2}{4\pi}\right)^2 J^R_{+,c}(x) + \left(1-\frac{n\lambda^2}{4\pi}\right)^2 J^R_{-,c}(x)\right]\delta(x-y)\,, \qquad (68)$$

$$\{J^L_{\pm,a}(x),\, J^R_{\mp,b}(y)\} = \mp\left(1-\left(\frac{n\lambda^2}{4\pi}\right)^2\right)\left(1\pm\frac{n\lambda^2}{4\pi}\right)t'_{ab}(x)\,\delta(x-y)\,, \qquad (69)$$

$$\{J^R_{\pm,a}(x),\, J^L_{\mp,b}(y)\} = \mp\left(1-\left(\frac{n\lambda^2}{4\pi}\right)^2\right)\left(1\mp\frac{n\lambda^2}{4\pi}\right)t'_{ba}(x)\,\delta(x-y)\,, \qquad (70)$$

and

$$\{J^L_{\pm,a}(x),\, t_{bc}(y)\} = -\left(1\pm\frac{n\lambda^2}{4\pi}\right)f^d_{ab}\, t_{dc}(x)\,\delta(x-y)\,, \qquad (71)$$

$$\{J^R_{\pm,a}(x),\, t_{bc}(y)\} = -\left(1\mp\frac{n\lambda^2}{4\pi}\right)f^d_{ac}\, t_{bd}(x)\,\delta(x-y)\,, \qquad (72)$$

together with eqn (52).

Note that all these Poisson bracket relations are consistent with the discrete symmetries of the model, which are built from

- the exchange $L \leftrightarrow R$ of left and right components, which originates from the transformation $g \to g^{-1}$ in G and must therefore be accompanied by a transposition of indices in t,

- the exchange $+ \leftrightarrow -$ of light-cone components, which originates from a parity transformation and must therefore be accompanied by an extra change of sign in spatial derivatives of t and of the delta function,

- the exchange $n \leftrightarrow -n$.

More specifically, the action of the model and the Poisson bracket relations are invariant under the combination of any two of these three symmetries:

a) $L \leftrightarrow R$ and $n \leftrightarrow -n$,

b) $+ \leftrightarrow -$ and $n \leftrightarrow -n$,

c) $L \leftrightarrow R$ and $+ \leftrightarrow -$.

Hence the case $n < 0$ can be reduced to the case $n > 0$ by exchanging either left and right components or light-cone components, so we may assume without loss of generality that $n > 0$. Next, recall that the critical value of the coupling constant λ is $\lambda = \sqrt{4\pi/n}$; here

$$J^L_+ = 2j^L_+ \quad,\quad J^L_- = 0 \quad, \qquad (73)$$
$$J^R_- = 2j^R_- \quad,\quad J^R_+ = 0 \quad. \qquad (74)$$

The vanishing of the two current components J^L_- and J^R_+ can be interpreted as constraints, and inspection of the commutation relations (64)–(72) (for $\lambda^2 = 4\pi/n$) shows that these constraints are first class, whereas the remaining commutation relations state that the non-vanishing current components J^L_+ and J^R_- satisfy a Kac-Moody algebra, differing only in the sign of the central extension, and commute with each other:

$$\{J^L_{+,a}(x),\, J^L_{+,b}(y)\} = -2f^c_{ab}J^L_{+,c}(x)\,\delta(x-y) + 8\eta_{ab}\,\delta'(x-y) \quad, \qquad (75)$$
$$\{J^R_{-,a}(x),\, J^R_{-,b}(y)\} = -2f^c_{ab}J^R_{-,c}(x)\,\delta(x-y) - 8\eta_{ab}\,\delta'(x-y) \quad, \qquad (76)$$
$$\{J^L_{+,a}(x),\, J^R_{-,b}(y)\} = 0 \quad. \qquad (77)$$

On the other hand, by taking the limit $\lambda \to 0$, or simply by setting $n = 0$, we recover the current algebra for the ordinary chiral model derived in ref. [3] from eqns (38)–(51) (which are equivalent to eqns (53)–(61) or (64)–(72)), together with eqn (52).

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