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ON THE AMPLITUDES FOR NON CRITICAL N=2 SUPERSTRINGS

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On the amplitudes for non critical N=2 superstrings.

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Abstract

We compute correlation functions in $N = 2$ non critical superstrings on the sphere. Our calculations are restrained to the $(s = 0)$ bulk amplitudes. We show that the four point function factorizes as a consequence of the non-critical kinematics, but differently from the $N = 0, 1$ cases no extra discrete state appears in the $\hat{c} \to 1^-$ limit.

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Critical N=2 strings have been recently considered by several authors[1,2,3]. Ooguri and Vafa[1] computed explicitly scattering amplitudes as, e.g., the four point function of the vertex operator

$$V(k) = \int d^4\theta d^2 z e^{ik\cdot\overline{X}(z)+i\overline{k}\cdot X(z)}$$

where X, $\overline{X}$, are the complex matter superfields and the on shell condition is given by $k \cdot \overline{k} = k_1\overline{k}_1 - k_2\overline{k}_2 = 0$. As it turns out, the four point function vanishes for the critical $N = 2$ superstring theory[1]. It seems to be true that the higher functions do also vanish in the critical theory. This result has been obtained as a consequence of the kinematics in 2+2 dimensional space-time. As a matter of fact, this vanishing was already expected, as argued by Ooguri and Vafa. Indeed, $N = 2$ superstrings are extremely simple string theories. Other string theories present an infinite tower of particles, which should appear as bound states in the critical string scattering amplitudes, as is the case of the Veneziano amplitude. However, in the $N = 2$ string, there is only a massless, scalar particle; the only way of obtaining consistency with the Veneziano amplitude and to avoid the infinite tower of states seems to be through the vanishing of the amplitude. This is actually what happens. There are several implications coming out of this vanishing of higher point functions as discussed in [1].

Nevertheless, even critical $N = 2$ strings are worthwhile studying. In fact, $N = 2$ theories are important objects in the study of integrable theories, and string vacua[4,5]. Moreover, there seems to be a strong relation between self duality in four dimensions and integrability[6], a fact that has extrapolated the barrier of dimensionality[7]. Finally, we should mention that there is a deep relation between integrable models and deformations of conformally invariant theories[8], which although very interesting will not concern us in the present work, but which might be important for $N = 2$ in order to understand the string vacuum.

Our present aim is to consider the non critical $N = 2$ string theory. This might be seen as a generalization of previous efforts to understand string theories away from criticality[9-18]. We will be actually concerned with a $N = 2$ matter supermultiplet with $\hat{c} \leq 1\,(c = 3\hat{c})$ in a super Coulomb gas representation conformally coupled to a $N = 2$ superliouville theory. However, as we shall see, the present case contains a number of new technical difficulties, which in part is due to the absence of the so called "barrier" in the central charge. In fact, both critical points coalesce, and the critical and non critical theories display a unique amalgamation of their properties, enhancing the difficulties in obtaining closed results.

The appearance of Liouville theory as a byproduct of the integration over matter fields in a gravity background (as well as its supersymmetric extension) is by now a very established issue[19-21]. In the case of $N = 2$ complex super Liouville (S_{SL}) theory interacting with a gravitational field we consider the action

$$
\begin{aligned}
S =& S_{SL} + S_M \\
S_{SL} =& \frac{1}{4\pi}\int d^2w\hat{E}\left[\int d^4\theta\left(\Phi\overline{\Phi} - Q\hat{Y}(\Phi+\overline{\Phi})\right) + \mu\left(\int d\theta^- d\overline{\theta}^- e^{\overline{\alpha}\Phi} + \int d\theta^+ d\overline{\theta}^+ e^{\alpha\overline{\Phi}}\right)\right] \\
S_M =& \frac{1}{4\pi}\int d^2w d^4\theta\hat{E}\left[X\overline{X} + 2i\alpha_0\hat{Y}(X+\overline{X})\right]
\end{aligned}
\tag{1}
$$

The superfields X, $\Phi(\overline{X}, \overline{\Phi})$ are chiral (antichiral) and we have explicitly:

$$
\begin{aligned}
X(z,\overline{z};\theta^-,\overline{\theta}^-) &= x(Z,\overline{Z}) + \psi_R(Z,\overline{Z})\theta^- + \psi_L(Z,\overline{Z})\overline{\theta}^- + G(Z,\overline{Z})\theta^-\overline{\theta}^- \\
\Phi(z,\overline{z};\theta^-,\overline{\theta}^-) &= \varphi(Z,\overline{Z}) + \xi_R(Z,\overline{Z})\theta^- + \xi_L(Z,\overline{Z})\overline{\theta}^- + F(Z,\overline{Z})\theta^-\overline{\theta}^-
\end{aligned}
\tag{2}
$$

where $(\theta^{\pm})^\dagger = \overline{\theta}^{\mp}$, $Z = z - \theta^+\theta^-$ and $\overline{Z} = \overline{z} - \overline{\theta}^+\overline{\theta}^-$. The quantity $\hat{Y}$ stands for the $N = 2$ supercurvature superfield and $\hat{E}$ for the superdeterminant of the superzweibein.

After setting $\mu = 0$ in S_{SL} we have the following expression for the last component of the super energy momentum tensor (holomorphic part):

$$
\begin{aligned}
T &= T_{SL} + T_M \\
T_{SL} &= -:\partial\overline{\varphi}\partial\varphi: + \frac{1}{4}:\overline{\xi}_R\partial\xi_R: + \frac{1}{4}:\xi_R\partial\overline{\xi}_R: - \frac{Q}{2}\partial^2(\varphi+\overline{\varphi}) \\
T_M &= -:\partial\overline{x}\partial x: + \frac{1}{4}:\overline{\psi}_R\partial\psi_R: + \frac{1}{4}:\psi_R\partial\overline{\psi}_R: + i\alpha_0\partial^2(x+\overline{x})
\end{aligned}
\tag{3}
$$

The first component of the super energy momentum tensor is given by the $U(1)$ current[5] which generates the $U(1)$ symmetry of $N = 2$ supersymmetric models. For the (holomorphic) part of this currents we have:

$$
\begin{aligned}
J &= J_{SL} + J_M \\
J_{SL} &= \frac{1}{4}:\overline{\xi}_R\xi_R: + \frac{Q}{2}\partial(\varphi - \overline{\varphi}) \\
J_M &= \frac{1}{4}:\overline{\psi}_R\psi_R: + i\alpha_0\partial(x - \overline{x}) \quad .
\end{aligned}
\tag{4}
$$

The propagators of the component fields can be read from the kinetic term of (1):

$$
\begin{aligned}
\langle x(z)\overline{x}(w)\rangle &= \langle\varphi(z)\overline{\varphi}(w)\rangle = \ln|z-w|^{-2} \\
\langle\psi_R(z)\overline{\psi}_R(w)\rangle &= \langle\xi_R(z)\overline{\xi}_R(w)\rangle = 2(z-w)^{-1} \\
\langle\psi_L(\overline{z})\overline{\psi}_L(\overline{w})\rangle &= \langle\xi_L(\overline{z})\overline{\xi}_L(\overline{w})\rangle = 2(\overline{z}-\overline{w})^{-1}
\end{aligned}
\tag{5}
$$

Following [22] we fix Q in (1) imposing the vanishing of the total central charge

$$
\begin{aligned}
c_T &= c_{SL} + c_M + c_{ghosts} = 0 \quad , \\
c_{SL} &= 3(1+2Q^2) \quad , \\
c_M &= 3\widehat{c}, \widehat{c} = 1 - 8\alpha_0^2 \quad , \\
c_{ghosts} &= -6 \quad ;
\end{aligned}
\tag{6}
$$

Thus we have:

$$
Q = 2|\alpha_0| \tag{7}
$$

where we chose Q to be real; this corresponds to a choice of phases, as one readily verifies.

The constants $\overline{\alpha}$ and α in eq. (1) can be fixed imposing that the operators $e^{\overline{\alpha}\Phi}$ and $e^{\alpha\overline{\Phi}}$ have dimension (1/2,1/2) (because of the double integration over the Grasmann variables):

$$
\begin{aligned}
\Delta\left(e^{\overline{\alpha}\Phi}\right) &= -\frac{\overline{\alpha}Q}{2} = \frac{1}{2} \\
\Delta\left(e^{\alpha\overline{\Phi}}\right) &= -\frac{\alpha\overline{Q}}{2} = \frac{1}{2}
\end{aligned}
\tag{8}
$$

$$
\alpha = \overline{\alpha} = -\frac{1}{\overline{Q}} = -\frac{1}{Q} \tag{9}
$$

Note that the operator $e^{\alpha\Phi}$ ($e^{\alpha\overline{\Phi}}$) is chiral (antichiral) since it satisfies the chirality condition $\Delta = +q(\Delta = -q)$, where the $U(1)$ charge q of a φ field is defined as usual from the short distance expansion

$$
J(w)\varphi(z) = \frac{q\varphi}{w-z} + \cdots \tag{10}
$$

It is easy to check that the values of α and $\overline{\alpha}$ in eq. (9) assure vanishing $U(1)$ charge for the action S_{SL} as required, with the basic assignments $q(d\theta^+) = 1/2 = -q(d\theta^-)$. Therefore the solution (9) is clearly a consistent one.

After having fixed the action, we must specify the vertex operators to calculate correlation functions. For comparison with the critical[1] case we shall be concentrated in vertex operators which are the analogous of the tachyon vertex operators in the $N = 0,1$ cases. However in the noncritical theories the operators must be dressed by gravity. In the $N = 2$ non-critical case the vertex operator reads:

$$
\begin{aligned}
V(k,\overline{k}) =& \int d^2z d^4\theta e^{i(k\overline{X}+\overline{k}X)+\beta\overline{\Phi}+\overline{\beta}\Phi} = \int d^2z e^{i(k\overline{x}+\overline{k}x)+\beta\overline{\varphi}+\overline{\beta}\varphi} \\
&\times\left[ik\partial\overline{x} + \beta\partial\overline{\varphi} - i\overline{k}\partial x - \overline{\beta}\partial\varphi - \overline{k}k\overline{\psi}_R\psi_R + \overline{\beta}\beta\overline{\xi}_R\xi_R + i\overline{\beta}k\overline{\psi}_R\xi_R - i\overline{k}\beta\psi_R\overline{\xi}_R\right] \\
&\times\left[ik\overline{\partial}\overline{x} + \beta\overline{\partial}\overline{\varphi} - i\overline{k}\overline{\partial}x - \overline{\beta}\overline{\partial}\varphi - \overline{k}k\overline{\psi}_L\psi_L + \overline{\beta}\beta\overline{\xi}_L\xi_L + i\overline{\beta}k\overline{\psi}_L\xi_L - i\overline{k}\beta\psi_L\overline{\xi}_L\right] \quad .
\end{aligned}
\tag{11}
$$

Notice that we have used on shell expressions (with $\mu = 0$) for the superfields $X(\overline{X})$:

$$\begin{aligned} X &= x(z,\overline{z}) + \psi_R(z)\theta^- + \psi_L(\overline{z})\overline{\theta}^- - \partial x\theta^+\theta^- - \overline{\partial}x\overline{\theta}^+\overline{\theta}^- \\ \overline{X} &= \overline{x}(z,\overline{z}) - \overline{\psi}_R(z)\theta^+ - \overline{\psi}_L(\overline{z})\overline{\theta}^+ + \partial\overline{x}\theta^+\theta^- + \overline{\partial}\overline{x}\overline{\theta}^+\overline{\theta}^- \end{aligned} \tag{12}$$

and analogously for Φ, $\overline{\Phi}$. In equation (11) the complex dressing β is fixed as a function of the complex momentum k by imposing that the vertex $V(k,\overline{k})$ be a dimensionless operator and its $U(1)$ charge vanishes. This amounts respectively to:

$$\begin{aligned} \Delta\left(e^{i(k\overline{X}+\overline{k}X)+\beta\overline{\phi}+\overline{\beta}\phi}\right) &= k(\overline{k}-2\alpha_0) + \overline{k}(k-2\alpha_0) - \overline{\beta}(\beta+Q) - \beta(\overline{\beta}+Q) = 0 \\ q\left(e^{i(k\overline{X}+\overline{k}X)+\beta\overline{\phi}+\overline{\beta}\phi}\right) &= Q(\beta-\overline{\beta}) + \alpha_0(k-\overline{k}) = 0 \quad . \end{aligned} \tag{13}$$

The first equation, with the help of the second one, fixes the real part of the dressing β (up to a sign):

$$\left(\frac{\beta+\overline{\beta}}{2}\right)_{\pm} = -\frac{Q}{2} \pm \left|\frac{k+\overline{k}}{2} - \alpha_0\right| \tag{14}$$

In this paper we assume the positive sign solution which is equivalent to positive energy particles. The equation (13) fixes the imaginary part of β.

It will be convenient later on to write the noncritical vertex in a form similar to the critical one[1]:

$$\begin{aligned} V(k,\overline{k}) &= \int d^2z d^4\theta e^{i[(k\cdot\overline{X})+(\overline{k}\cdot X)]} = \int d^2z e^{i(k\cdot\overline{x}+\overline{k}\cdot x)} \\ &\times [ik\cdot\overline{\partial}\overline{x} - i\overline{k}\cdot\partial x - (k\cdot\overline{\psi}_R)(\overline{k}\cdot\psi_R)][ik\cdot\overline{\partial}\overline{x} - i\overline{k}\cdot\overline{\partial}x - (k\cdot\overline{\psi}_L)(\overline{k}\cdot\psi_L)] \end{aligned} \tag{15}$$

where the scalar product is defined by $a\cdot b = a^1b^1 + a^2b^2$ and* $k = (k, -i\beta)$, $\overline{k} = (\overline{k}, -i\overline{\beta})$, $X = (X,\varphi)$, $\overline{X} = (\overline{X},\overline{\varphi})$, $x = (x,\varphi)$, $\overline{x} = (\overline{x},\overline{\varphi})$, $\psi_{R(L)} = (\psi_{R(L)}, \xi_{R(L)})$, $\overline{\psi}_{R(L)} = (\overline{\psi}_{R(L)}, \overline{\xi}_{R(L)})$. Notice that the second component of the vector $\overline{k}$ is not the complex conjugate of the second component of the vector k in the non critical case

We are now ready to compute correlation functions

$$\langle V_{k_1}\cdots V_{k_n}\rangle = \prod_{i=1}^{n}\int d^2z_i d^4\theta_i \left\langle \prod_{i=1}^{N} e^{i(k_i\overline{X}_i+\overline{k}_iX_i)+\beta_i\overline{\phi}_i+\overline{\beta}_i\phi_i}\right\rangle_{S_M+S_{SL}} \tag{16}$$

* We use the same symbol for the two component vector and for its first component; there will be no room for confusion since the two component vector will only appear inside scalar products, denoted by a dot.

The first step is to integrate over the two zero modes x_0^1, x_0^2 of the first component of the matter superfield ($x = x^1 + ix^2$). The result gives the conservation of the real and imaginary parts of the momenta k, both are encoded in the following formula:

$$\sum_{i=1}^{n} k_i = 2\alpha_0 \tag{17}$$

The next step is the integration over the Liouville zero modes[9,10] φ_0^1, φ_0^2 ($\varphi = \varphi^1 + i\varphi^2$) this is more

delicate in the $N = 2$ case. If we naively integrate over φ_0^1 and φ_0^2 in eq.(16) we have a divergent result. It is not clear[23] how this divergence should be regularized. We opt for making a Wick rotation in the zero modes such that φ_0 and $\overline{\varphi}_0$ become real. After integrating over $\varphi_0, \overline{\varphi}_0$ we have:

$$\begin{aligned} \mathcal{A}_n &= \frac{\mu^{s+\overline{s}}}{\alpha^2}\Gamma(-s)\Gamma(-\overline{s})\int\prod_{i=1}^{n} d^2z_i d^4\theta_i \left\langle \prod_{j=1}^{n} e^{i(k_j\overline{X}_j+\overline{k}_jX_j)+\beta_j\overline{\phi}_j+\overline{\beta}_j\phi_j}\right. \\ &\times \left.\left(\int d^2\omega d^2\overline{\theta} e^{\alpha\overline{\phi}}\right)^{\overline{s}}\left(\int d^2\eta d^2\theta e^{\alpha\phi}\right)^{s}\right\rangle \end{aligned} \tag{18}$$

where $s = -\frac{1}{\alpha}(\sum\beta_i + Q)$, and $\overline{s} = -\frac{1}{\alpha}(\sum\overline{\beta}_i + \overline{Q})$. Although our regularization is rather ad'hoc we believe that our results for $s = 0 = \overline{s}$ bulk amplitudes are independent of this procedure. From now on we only consider $s = \overline{s} = 0$ bulk amplitudes:

$$\sum_{1}^{n}\beta_i + Q = 0 = \sum_{1}^{n}\overline{\beta}_i + Q \tag{19}$$

We start by looking at the $n = 3$ point function. After fixing the residual $OSP(2,2)$ symmetry choosing $\theta_1^{(\pm)} = \theta_3^{(\pm)} = 0$ and $z_1 = \infty$, , $z_2 = 1$, $z_3 = 0$ we have:

$$\begin{aligned} \mathcal{A}_3 &= \frac{(\ln\mu)^2}{\alpha^2}\left\langle e^{ik_3\cdot x(0)}e^{ik_2\cdot x(1)}[ik\cdot\overline{\partial}\overline{x} - i\overline{k}\cdot\partial x - (k\cdot\overline{\psi}_R)(\overline{k}\cdot\psi_R)]\right. \\ &\left.\times[ik\cdot\overline{\partial}\overline{x} - i\overline{k}\cdot\overline{\partial}x - (k\cdot\overline{\psi}_L)(\overline{k}\cdot\psi_L)]\right\rangle_{S(\mu=0)} \end{aligned}$$

$$\mathcal{A}_3 = \frac{(\ln\mu)^2}{\alpha^2}(c_{23})^2 \tag{20}$$

where

$$\begin{aligned} c_{ij} &= k_i\cdot\overline{k}_j - \overline{k}_i\cdot k_j \\ &= k_i\overline{k}_j - \overline{k}_ik_j - \beta_i\overline{\beta}_j + \overline{\beta}_i\beta_j \end{aligned} \tag{21}$$

and $(\ln\mu)^2$ = finite part of $\left(\lim_{\bar{s},s\to 0}\mu^{s+\bar{s}}\Gamma(-s)\Gamma(-\bar{s})\right)$. In order to rewrite $\mathcal{A}_3$ in a more sugestive form we need some kinematics. First of all it is easy to show (using (13),(17) and (19)) that

$$\sum_{j=1}^{n} c_{ij} = 2\alpha_0(k_i - \bar{k}_i) + (\beta_i - \bar{\beta}_i)Q = 0 \tag{22}$$

The vanishing of $\sum_{j=1}^{n} c_{ij}$ holds in the critical[1] case as a consequence of the momentum conservation and the on shell condition $k\cdot\bar{k} = k_1\bar{k}_1 - k_2\bar{k}_2 = 0$. It is remarkable that (22) holds also in the non critical case as a consequence of the zero $U(1)$ current condition (13).

We assume* from now on that $\alpha_0 < 0$ in this case we have from (13) and (14)

$$\beta(k) = \begin{cases} k & \text{, if } \Re e k = \frac{k+\bar{k}}{2} > \alpha_0 \\ 2\alpha_0 - \bar{k} & \text{, if } \Re e k < \alpha_0 \end{cases} \tag{23}$$

therefore

$$k\cdot\bar{k} = k\bar{k} - \beta\bar{\beta} = \begin{cases} 0 & \text{, if } \Re e k > \alpha_0 \\ 4\alpha_0(\Re e k - \alpha_0) & \text{, if } \Re e k < \alpha_0 \end{cases} \tag{24}$$

Using all these kinematic relations we may write $\mathcal{A}^3$ from (20) in the region $\Re e k_2$, $\Re e k_3 < \alpha_0$, $\Re e k_1 > \alpha_0$ in a factorized form:

$$\mathcal{A}_3 = \frac{(\ln\mu)^2}{\alpha^2}\frac{(\Im m k_1)^2}{\alpha_0^2}(k_2\cdot\bar{k}_2)(k_3\cdot\bar{k}_3) \tag{25}$$

where $\Im m k = (-i)^{\frac{(k-\bar{k})}{2}}$. The amplitude vanishes for any other kinematic region, where at least two momenta satisfy $Rek > \alpha_0$ and are therefore "on shell" $(k\cdot\bar{k} = 0)$ in the critical sense. It should be stressed that the amplitude $\mathcal{A}_3$ in the critical case[1] has the same form (20) but it cannot be written in a factorized form as in (25).

Now we calculate the four-point amplitude $\mathcal{A}_4$. Fixing $\theta_1^{(\pm)} = \theta_4^{(\pm)} = 0$ and $z_1 = \infty$, $z_2 = 1$, $z_3 = z$, $z_4 = 0$ we have , after using (22), basically the same expression as in the critical case:

$$\mathcal{A}_4 = \frac{(\ln\mu)^2}{16\alpha^2}\int d^2z|z|^{-s}|1-z|^{-t}\left(\frac{t(t+2)}{(1-z)^2} + \frac{4c_{12}c_{34}}{z} + \frac{4c_{23}c_{41}}{1-z}\right)\times(\text{ "h. c." }) \tag{26}$$

where** $s = -2s_{34}$, $t = -2s_{23}$ and $s_{ij} = k_i\cdot\bar{k}_j + \bar{k}_i\cdot k_j$. The hermitian conjugated part above corresponds to the previous term inside the brackets with $\bar{z}$ instead of

* The calculations for $\alpha_0 > 0$ are completely analogous.

** Our definition of s and t correspond to twice of ref.[1] because their propagator correspond to half of ours (see (5))

z. Note that it is not really the hermitian conjugated expression since $\bar{k}_i$ is not the complex conjugated of k_i as we stressed before.

After performing the integrals in (26) using formulas of ref.[1] and making algebraic manipulations which are consequence of kinematic relations common to the critical and noncritical cases we have:

$$\mathcal{A}_4 = -\frac{\pi(\ln\mu)^2}{\alpha^2}F^2\Delta(s_{34})\Delta(s_{14})\Delta(s_{24}) \tag{27}$$

where

$$F = [(k_1\cdot\bar{k}_2)(k_2\cdot\bar{k}_3)(k_3\cdot\bar{k}_1) + \text{ "h. c."}] \tag{28}$$

and $\Delta(x) = \Gamma(x)/\Gamma(1-x)$.

The expression (27) is essentially the same one derived in critical case, the difference now comes from the fact that after fixing the kinematic region $\Re e k_1$, $\Re e k_2$, $\Re e k_3 < \alpha_0$, $\Re e k_4 > \alpha_0$ we have, after a long algebra, a very simple expression for F:

$$F = -\frac{((\Im m k_4)^2 + \alpha_0^2)}{4\alpha_0^2}\prod_{i=1}^{3}(k_i\cdot\bar{k}_i). \tag{29}$$

In the above kinematic region we also obtain $s_{i4} = -(k_i\cdot\bar{k}_i)$, therefore we can finally write $\mathcal{A}_4$ in a factorized form:

$$\mathcal{A}_4 = \frac{\pi(\ln\mu)^2}{16\alpha^2}\frac{((\Im m k_4)^2 + \alpha_0^2)^2}{\alpha_0^4}\prod_{1}^{3}\Delta(1 - k_i\cdot\bar{k}_i). \tag{30}$$

As in the case of the 3-point function ($\mathcal{A}_3$), it can be shown that $\mathcal{A}_4$ vanishes in any kinematic region where at least two momenta satisfy $\Re e k > \alpha_0$.

In the critical limit $\hat{c}\to 1\,(\alpha_0\to 0)$ we have $\Delta(1 - k_i\bar{k}_i)\sim\alpha_0$ thus,

$$\mathcal{A}_4(\alpha_0\to 0) \sim \frac{(\Im m k_4)^4}{\alpha_0\alpha^2}(\ln\mu)^2. \tag{31}$$

If we absorb the factor $1/\alpha^2\,(= 4\alpha_0^2)$ in the measure of the path integral the amplitude $\mathcal{A}_4$ diverges like $1/\alpha_0$, otherwise the amplitude vanishes $\mathcal{A}_4\sim\alpha_0$ as in the critical case[1]. It should be noticed, however, that the factor $1/\alpha^2$ (which comes from the double zero mode integrals) must be absent in $\mathcal{A}_3$ (see (20)) if we want to obtain a non-vanishing result in the $\alpha_0\to 0$ limit, as in the critical case. Thus we can not obtain both $\mathcal{A}_3$ and $\mathcal{A}_4$ of the critical case in the $\alpha_0\to 0$ limit (see conclusion).

For $\hat{c} < 1$ the interesting models are the minimal ones and in these cases it is easy to show that the functions $\Delta(1 - k_i\cdot\bar{k}_i)$ have no poles (or zeroes) thus, they can be simply absorbed in the definition of the vertices $V(k,\bar{k})$ exactly as in the

$N = 0, 1$ cases. Actually the factor $\Delta(1-k_i\bar{k}_i) = \Delta(1+\beta\bar{\beta}-k\bar{k})$ corresponds to the factors[12] $\Delta\left(\frac{1}{2}(1+\beta^2-k^2)\right)$ and[13] $\Delta\left(\frac{1}{2}(\beta^2-k^2)\right)$ of the $N = 1$ and $N = 0$ cases respectively, in the sense that all these factors become $\Delta(1) = 0$ for configurations with zero energy ($E = \Re e\beta + \frac{Q}{2} = 0$) which sho ws the decoupling of such states[24].

Conclusions

We have calculated the three and four point ($s = 0$) bulk amplitude, in a non-critical $N = 2$ superstring consisting of a $N = 2$ matter supermultiplet with central charge $\hat{c} \leq 1\,(c = 3\hat{c})$ conformally coupled to an $N = 2$ super Liouville theory. We have shown that both amplitudes may be written ,in a certain kinematic region, in a factorized form. For other kinematic regions the amplitudes vanish. Moreover after a suitable renormalization of the measure of the path integral we get a the three-point amplitude which is finite[1] in the limit $\alpha_0 \to 0\,(\hat{c} \to 1^-)$, as in the critical case. There is, however, an amazing difference with respect to the critical string when we look at the four-point amplitude in the same limit, namely, we get a divergent result ($\mathcal{A}_4 \sim 1/\alpha_0$) rather than a vanishing one[1]. The difference with the critical case can be explained as follows, the prefactor F in (29), which vanishes identically in the critical case[1], goes to zero in the critical limit ($F \sim \alpha_0$) but its vanishing is supressed by the poles of the functions $\Delta(-k_i \cdot \bar{k}_i)\,(\Delta(-k_i \cdot \bar{k}_i) \sim 1/\alpha_0)$ which usually correspond to intermediate states*, such poles cancel in the critical case but they show up in the non critical one as a direct consequence of the non analytic structure of the dispersion relation (14) which permited to fix completelly, in a certain kinematic region, the real part of the momenta of one of the scattered particles (in our case $\Re e k_4 = -\alpha_0$). This means that there may not exist a smooth limit from the non critical to the critical case. We should also mention that it is possible to redefine the vertices $V(k, \bar{k})$ and the path integral measure by powers of α_0 such that the amplitudes become finite in the $\alpha_0 \to 0$ limit. In order to conclude the discussion about the $\hat{c} \to 1^-$ limit we remark that whatever is the correct interpretation of $\mathcal{A}_4$ we do not have extra discrete states as in the $N = 0, 1$ cases and this is expected since the spectrum of critical $N = 2$ string is finite.

For $\hat{c} < 1$ the effect of the functions $\Delta(-k_i \cdot \bar{k}_i)$ is as mild as in the $N = 0, 1$ cases and the $s = 0$ amplitudes are basically given by the factor $(\ln \mu)^2$.

Several aspects of our results are still unclear and a more carefull analysis is needed, wich would imply the calculation of higher point amplitudes as well as $s \neq 0$ correlators, this is under progress.

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* Since α_0 works like an infrared cut-off in non-critical calculations[13] we might even speculate that such poles correspond to the exchange of the massless particle present in the critical theory. The non vanishing contribution of those poles in the non critical case might be attributed to the non trivial kinematics.

Reference

[1] H. Ooguri, C. Vafa, Nucl. Phys. **B361** (1991)469, **B367** (1991)83.

[2] A. Giveon and A. Shapere, Cornel and Princeton preprints CLNS-92/1139, IASSNS-HEP-92/14.

[3] M. Li, Univ. California, Santa Barbara, UCSBTH-92/14

[4] C. Vafa, at the Simp. on Fields, Strings and Quantum gravity, Beijing, 1989, and Summer School in High Energy Physics and Cosmology, ICTP, 1989, 1990.

[5] P. Fendley, S. D. Mathur, C. Vafa and N. P. Warner, Phys. Lett. **B243** (1990)257.

[6] M. K. Prasad, A. Sinha, L. L. Chau Wang, Phys. Rev. Lett. **43** (1979)750.

[7] E. Witten Nucl. Phys **B266** (1986)255; J. Avan, H. J. de Vega and J. M. Maillet Phys. Lett. **171B** (1986)255; E. Abdalla, M. Forger and M. Jacques Nucl. Phys. **B307** (1988)198.

[8] V. A. Fateev, Int. J. Mod. Phys. **A12** (1991)2109; P. Fendley and K. Intriligator HUPT-91/A067.

[9] A. Gupta, S. P. Trivedi and M. B. Wise, Nucl. Phys. **B340** (1990)475.

[10] M. Goulian and M. Li, Phys. Rev. Lett. **66** (1991)2051.

[11] Vl. S. Dotsenko Mod. Phys. Lett **A6** (1991)3601.

[12] E. Abdalla, M. C. B. Abdalla, D. Dalmazi, K. Harada, Phys. Rev. Lett. **68** (1992) 1641; Int. J. Mod. Phys. **A** to appear.

[13] P. di Francesco and D. Kutasov, Nucl Phys. **B375**(1992)119.

[14] I. Klebanov, Priceton Preprint PUPT-1271, 1991.

[15] K. Aoki, E. d'Hoker, Mod. Phys. Lett. **A7** (1992) 235,333.

[16] L. Alvarez-Gaumé, Ph. Zaugg, Phys. Lett. **B273** (1991)81

[17] L. Alvarez-Gaumé and J. L. Mañez, Mod. Phys. Lett. **A6** (1991)2039; L. Alvarez-Gaumé, H. Itoyama, J. L. Mañez and A. Zadra, CERN prep, 1992

[18] G. Moore, N. Seiberg, M. Staudacher, Nucl. Phys. **B362**(1991)665

[19] A. M. Polyakov Phys. Lett. **B103** (1981)207.

[20] A. M. Polyakov Phys. Lett. **B103** (1981)211.

[21] O. Alvarez Nucl. Phys. **B216** (1986)125;*ibid* **B238**(1984) 61; E. Abdalla, M. C. B. Abdalla and K. D. Rothe, Non perturbative methods in 2 dimensional quantum, World Scientific Publising, 1991.

[22] J. Distler, Z.Hlousek, H. Kawai, Int. J. Mod. Phys. **A5** (1990)391.

[23] N. Seiberg and D. Kutasov, Phys. Lett. **B251**(1990)67.

[24] N. Seiberg, Lecture at 1990 Yukawa Int. Sem. Common Trends in Math. and Quantum Field Theory, and Cargese meeting Random Surfaces, Quantum Gravity and Strings, May 27, June 2, 1990; J. Polchinski, Strings '90 Conference, College Station, TX, Mar 12-17, 1990, Nucl. Phys. **B357**(1991)241.