Gravitational Waves from Mini Black Holes Binaries:
General Relativity and Schrödinger-Newton Equation

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Abstract.

This is a *"divertissement"* paper written to graduate and postgraduate students of Physics. We estimate gravitational waves emitted by *mini* non-charged black holes binaries (**μBHb**) using general relativity and a quantum mechanical approach according to the Schrödinger-Newton equation.
Key words. mini black hole binary; gravitational waves; *gravitational quantum effects*.
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(I) Introduction .

We estimate gravitational waves emitted by *mini* non-charged black holes binaries (**μBHb**) with the quantum mechanical approach based in an *hypothetical* non-relativistic quantum mechanical Schrödinger-Newton equation. In our analysis are also taken into account classical mechanics[1], classical electrodynamics,[2] quantum mechanics (**QM**),[3,4] special relativity **(SR)** and general relativity(**GR**).[5-7] In **Section 1** are shown some parameters of mini black holes (**μBH**). In **Section 2** is seen how to estimate with **GR** the gravitational luminosity L_{GW} = dE/dt of a black hole binary (**BHb)**. In **Section 3** we have assumed that **μBHb** obeys a Schrödinger-Newton equation. In this way, we have calculated the gravitational energy L_{GW} emitted by the **μBHb** using an "hybrid" **GR** & **QM** approach. As our model is **controversial**, are presented in **Section Φ, Section 4, Conclusions & Discussions and Appendix A, B and C** many comments and discussions involving our approach

(1) Few Parameters of mini black holes.

In **Figure 1** is shown a mini black hole binary (**μBHb**) composed by two non-charged **mini** black holes masses $\mu = m_1 = m_2$.[5-7]

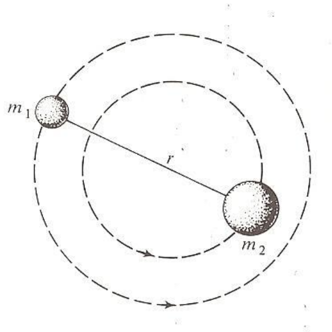

Figure 1. Mini black holes binary (**μBHb**).

The masses **μ,** according to the **GR**[5] can be arbitrarily small, however, its minimum mass is given by Planck mass[7,8] $\mu_{min} = \mathbf{M_P} = (\hbar c/G)^{1/2}$. Associated with this mass we have the Plank length $\boldsymbol{\ell_P} = \hbar/cM_P$. These values are shown below by **Eqs.(1.1)** and **(1.2).** The **BH** Schwarzschild radius[9] $r_{\underline{s}} = 2G\mu/c^2$ and its lifetime[10] $\boldsymbol{\tau_H} = 5120\pi G^2\mu^3/(\hbar c^4)$, due to the Hawking radiation, are shown by **Eqs.(1.3)** and **(1.4)**. The metric tensor component $g_{oo}(r)$, given by **Eq.(1.5),** is written in terms of the constants c, G and ħ, in the **MKS** system,

$$\mathbf{M_P} = (\hbar c/G)^{1/2} \sim 2\ 10^{-8} \quad (\mathrm{Kg}) \qquad (1.1),$$

$$\boldsymbol{\ell_P} = \hbar/cM_P \sim 1.616\ 10^{-35} \quad (\mathrm{m}) \qquad (1.2)$$

$$\mathbf{r_{\underline{s}}} = 2G\mu/c^2 \sim 1.5\ 10^{-27}\ \mu \quad (\mathrm{m}) \qquad (1.3),$$

$$\boldsymbol{\tau_H} = 5120\pi G^2\mu^3/(\hbar c^4) \sim 4\ 10^{-18}\ \mu^3 \quad (\mathrm{s}) \qquad (1.4),$$

$$g_{oo}(r) = -1 - 2G\mu/rc^2 \qquad (1.5),$$

(2)Gravitational luminosity of a BHb according to GR.

Gravitational waves emitted by a *black hole binary* (**BHb**), with black holes with total mass $M_+ = M_1 + M_2 \sim 20 - 30$ solar masses, have been recently detected by Abbott et al.[11,12] The unstable **BHb** motion can be divided into three stages:[11-13] "inspiral", "merger" (or "plunge") and "ringdown". During this motion the **BHb** emits GW. The "inspiral" is the first stage of the **BHb** life which resembles a gradually shrinking orbit; the emitted GW are weak when **BH** are distant from each other. During the **"inspiral motion"** with $M_1 \approx M_2 \approx M^*$ the gravitational luminosity L_{GW} is given by, [5,13-16]

$$L_{GW} = dE/dt \approx (8G/5c^5)\, M^{*2}\, r^4\, \omega^6 \qquad (2.1),$$

where r is distance between the black holes and **ω** is the **orbital rotational frequency** described by Kepler´s law[1,5] $r(t)^3\omega(t)^2 = GM_+ \approx 2GM^*$.

The "**spiral time**" τ [5,16] of the **BHb** can be estimated taking into account the total mechanical energy $E = I\omega^2/2 - GM^{*2}/r$ and using the "virial" theorem,[1] getting $E = - GM^{*2}/2r$. Taking this last result and **Eq.(2.1)** we have[5]

$$dr/dt = - (128/5c^5)\, G^3M^{*3}/r^3 \qquad \text{that is,}$$

$$r^3\, dr/dt = (1/4)d(r^4)/dt = -(128/5c^5)\, G^3M^{*3} \qquad (2.2).$$

Integrating **Eq.(2.2)** from r_o up to $2r_s$, defined by **Eq.(1.3)**, we have

$$r_o^4 = (2r_s)^4 - (128/5c^5)\, G^3M^{*3}\,\tau \qquad (2.3),$$

where $\boldsymbol{\tau}$, also called **"time to fall"** from a generic orbit $r = r_o$ to the **closest distance** $2r_s$ between two the **BH**. It is given by :

$$\tau = [5c^5/(128\, G^3M^{*3})]\,(r_o^4 - 16r_s^4) \qquad (2.4).$$

According to LIGO observations from a **BHb**,[11,12] known as GW150914 and GW151226 events, were measured the GW frequencies, in the range 30 -500 Hz, luminosities $L_{GW}(t)$ and decay times τ. Assuming that **BHb** were composed by masses $M^* \sim 10^{30}$ kg these results are in good agreement with **GR** estimations. The GW emitted by a **μBHb** could be evaluated with our equations[13] simply replacing M^* by μ.

(3)μBHb described by Schrödinger-Newton Equation.

According to **Section 1**, **μBHb** systems have **microscopic dimensions**. In addition to this, let us assume that in these systems interaction processes have microscopic energies, compatible with their dimensions. So, we *postulate* that **μBHb** can be described a **Schrödinger-Newton equation[17]**

$$H =\{(\hbar^2/2\mu)\Delta - G\mu^2/r\}\Psi(r,\theta,\varphi) = E\Psi(r,\theta,\varphi) \qquad (3.1),$$

Solving **Eq.(3.1)**[3,4] the **μBHb** *gravitational energies* $(E_g)_n$ would be given by

$$(E_g)_n = - \Theta_{grav}/n^2, \qquad (3.2),$$

where n = 1,2,3,...and

$$\Theta_{grav} = (\mu/2)(G\mu^2)^2/2\hbar^2 = G^2\mu^5/4\hbar^2 \qquad (3.3).$$

On the other hand, for the **hydrogen atom** (**HA**) we have,[3,4]

$$(E_{elect})_n = - \Theta_{elect}/n^2 \qquad (3.4),$$

where $\Theta_{elect} = m_e e^4/2\hbar^2$ and m_e = electron mass. That is,[3,4]

$$\Theta_{electr} = 13.6 \text{ eV} \sim 10^{-18} \text{ J} \qquad (3.5).$$

For the **HA** the "**electromagnetic Bohr radius**" $(a_o)_{elect}$ is given by[3,4]

$$(a_o)_{elect} = \hbar^2/me^2 \sim 0.5\ 10^{-10}\ \text{m} \qquad (3.7).$$

Taking into account **Eq.(3.1),** the "**gravitational Bohr radius**" $(a_o)_g$ is given by,

$$(a_o)_g = \hbar^2/G^2\mu^3 \qquad (3.8).$$

The electronic orbit radius $(r_n)_{elect}$ are given by

$$(r_n)_{elect} = n^2(a_o)_{elect} = n^2\ (\hbar^2/me^2) \sim n^2\ 0.5\ 10^{-10}\ \ (\text{m}) \qquad (3.9)$$

and "gravitational" radius $(r_n)_g$ are given by

$$(r_n)_g = n^2(a_o)_g = n^2(\hbar^2/G^2\mu^3) \sim 2.52\ 10^{-48}(n^2/\mu^3)\ (\text{m}) \qquad (3.10).$$

(3.1) μBHb Stability.

As well known, the **Hydrogen Atom (HA)** ground state with n = 1 is stable.[3,4] In this state the atomic radius $r \sim 10^{-10}$ m that is larger than the nuclear radius $\sim 10^{-15}$ m. So, the electron can be thought as moving in an orbit very far from nucleus and that it is an essential to the atomic stability. Of course, no **BHh** is stable.[11,13] Our "quantum" **μBHb** in higher states n decays to small levels. If we assume that the **μBHb** ground state n = 1 cannot be stable its "*binary Bohr radius"* $(a_o)_g = \hbar^2/G^2\mu^3$ must be must bigger than the Schwarzschild radius[9] $r_s = 2G\mu/c^2$. This would occur because inside the sphere with radius $r_s = 2G\mu/c^2 \sim 1.5\ 10^{-27}\ \mu$ (m) would be a "contact" between the mini black holes. In this way, our **μBHb** quantum model would be "consistent" only if its radius r obeys the condition $r \geq (a_o)_g$. That is, when $\hbar^2/G^2\mu^3 > 2G\mu/c^2$, or

$$\mu < (\hbar/c)^{1/2}\ G^{-3/2} \sim 10^{-5}\ \text{kg} \qquad (3.11).$$

(3.2) GW emitted by a μBHb.

In what follows we will analyze only **μBHb** obeying the condition $\boldsymbol{\mu < 10^{-5}}$ **kg**. Thus, as done in preceding papers,[11-14]we divide the **μBHb** motion in three stages: **"inspiral", "merger"** (or **"plunge")** and **"ringdown".** During this motion the binary is emitting GW. The **"inspiral**" is the first stage which resembles a gradually shrinking orbit and take a longer time; the emitted GW are weak because the black holes are very distant. As the **BHb** orbit shrinks, the speeds of the mini black holes increase, and the intensity of GW increases. When the black holes are close the orbit shrinks very quickly and they reach extremely high velocities. This is followed by a plunging orbit and they will **"merge"** once they are close enough. At this moment the GW amplitude reaches its peak. Once merged, the single hole settles down to a stable form, via a stage called **"ringdown",** where any distortion in the shape is dissipated as more GW.[11-14]

Remembering that the energies $(E_g)_n = -(G^2\mu^3/4\hbar^2)/n^2$, the emitted gravitational energies **ħω,** between two states with n and n+1, according to **Eqs.(3.2)** and **(3.3),** are given by,

$$\hbar\omega = (E_g)_{n+1} - (E_g)_n = -(G^2\mu^5/4\hbar^2)[1/(n+1)^2 - 1/n^2] \qquad (3.12),$$

showing that the emitted GW can have an *infinity* of values.

(3.3)Inspiral motion.

In the **inspiral motion**,[11-14] the mini black holes would be very distant, that is, when $r >> (a_o)_g$ and $g_{oo}(r) = -1- 2G\mu/c^2 r \approx -1$, that is,

$$r >> 2G\mu/c^2 \qquad (3.13).$$

As $(r_n)_g = n^2(a_o)_g$ we must have when $n >> 1$ implying that **Eq.(3.12)** can be given by

$$\hbar\omega = E^g{}_{n+1} - E^g{}_n \approx -(G^2\mu^5/4\hbar^2)(1/n^4) = -10^{46}\mu^5/n^4 \ \ (J) \qquad (3.14).$$

(3.4) Gravitational Luminosity from a μBHb.

For $r_n \geq 10^{-30}$ m, for instance, we have, using **Eq.(1.5),** $g_{oo}(r) \sim -1$, showing that the gravitational distortions of the metric are very small.[5] As, $n > 1$ and $r >> r_s$ gravitational relativistic effects will be neglected . For sufficiently large n the **μBHb** will be in the **inspiral motion** that is, with $(r_n) > r_s$ and with the energy values $(E_g)_n$ given by,

$$(E_g)_n = -(G^2/4\hbar^2)\,\mu^5/n^2 \approx -10^{47}\,\mu^5/n^2 \quad (J). \qquad (3.15)$$

The emitted energies $\hbar\omega$ in the transitions $n \rightarrow n + 1$ are given by

$$\hbar\omega = (E_g)_{n+1} - (E_g)_n = -\mu^5 10^{47}\,[1/(n+1)^2 - 1/n^2\,]\,J \approx -\mu^5 10^{47}/n^4 \ \ (J) \qquad (3.16).$$

Taking, for instance, $\boldsymbol{\mu = 10^{-12}}$ **kg** we verify that

$$\omega \approx 10^{-13}/n^4 \ J \approx 10/n^4 \quad (rad/s) \qquad (3.17).$$

On the other hand, if $\boldsymbol{\mu = 10^{-11}}$**kg** we have, instead of (3.17)

$$\omega \approx 10^6/n^4 \quad rad/s \qquad (3.18),$$

which means that, for large n like n ~10 we obtain

$$\boldsymbol{\omega \approx 100 \quad rad/s.}$$

Note that recent detected GW[11,12] emitted by a **BHb,** when M* ~ 30 solar mass, had frequencies $\boldsymbol{\omega \sim (150\pi - - -170\pi)}$ **rad/s.**

Thus, **μBHb** mini black holes with masses $\mu < 10^{-5}$ Kg could emit GW with frequencies **similar** to those emitted by massive **BHb.**

(Φ) Comments on Schrödinger-Newton Quantization

Let us consider GW with energies $\hbar\omega = (E_g)_{n+1} - (E_g)_n$, given by **Eq. (3.12),** emitted in transitions $n \rightarrow n + 1$. To occurs this emission , we believe that is necessary to have *some kind* of interaction *W*(t)(*what kind?)* that induces transitions between the quantum states $n \rightarrow n + 1$. It can be estimated taking into account a *perturbation theory* derived from Schrödinger-Newton equation. Let us write *W*(t) harmonically dependent on the time[4]

$$W^{\pm}(t) = w^{\pm} \exp[\pm i\omega t] \qquad (\Phi.1),$$

where $w^{\pm}$ is time independent. In the Schrödinger theory it can be shown[4] that the transition probability $m \rightarrow n$ per unit of time $P^{\pm}_{nm}$ is given by

$$P^{\pm}_{nm} = (2\pi/\hbar)\, |\langle n \,|\, w^{\pm} \,|\, m \rangle|^2\, \delta(E_n - E_m \pm \hbar\omega) \qquad (\Phi.2),$$

where the + and - correspond to the signs in the exponential in **Eq.(Φ.1).** Thus, under this perturbation, transitions take place to states with energies satisfying the condition $E_m = E_n \pm \hbar\omega$. If the perturbation is of the form $W^{+}(t) = w^{+} \exp(i\omega t)$ the system loses an energy $\hbar\omega$ (energy is emitted), since $E_n = E_m - \hbar\omega$ in the transition, while if it is of the form $W^{-}(t) = w^{-} \exp(-i\omega t)$ it gains an energy $\hbar\omega$, since $E_n = E_m + \hbar\omega$. Our **main problem** is to determine the function $W^{\pm}(t)$. The gravitational "luminosity" $(L_{GW})_{nm}$ in the *inspiral* stage would be given by $(L_{GW})_{nm} = \hbar\omega\, P^{+}_{nm}$, for very large quantum numbers.

Before to propose a model to obtain $W^{+}(t)$ let us remember that according to Bohr correspondence principle (CP)[3] for very large quantum numbers, classical and quantum physics are expected to give the same answer, at least in average. The probabilistic interpretation of the phenomenon obtained with the Schrödinger´s equation will give, in average the same results obtained by classical laws. Ehrenfest, for instance, showed that Newton's laws hold on average: the quantum statistical expectation value of the position and momentum obey Newton's laws. Thus, we hope that in the **μBHb** inspiral stage properties estimations given by the "classical" GR and QM laws agree *in average*. In addition, as seen in **Appendix B** and **C,** in Classical Electrodynamics the luminosities L_ω , emitted by **dipolar** and **quadrupolar** radiation are given, respectively, by

$$L_\omega = dE/dt = (ck^4/3)\, |\mathbf{D}|^2 = (\omega^4/3c^3)\, |\mathbf{D}|^2 \qquad \text{and} \qquad (\Phi.3)$$
$$L_\omega = dE/dt = (\omega^6/360c^5) \sum\nolimits_{\alpha\beta} |Q_{\alpha\beta}|^2 \,.$$

In Quantum Electrodynamics they are given, by $L_\omega = (4\omega^4/3c^3)\, |\boldsymbol{D}_{nm}|^2$ and $L_\omega \approx (\omega^6/2\pi c^5)|Q_{nm}|^2$, respectively, where $\omega = \omega_{nm}$, $\boldsymbol{D}_{nm} = \langle n\, |\mathbf{D}|\, m \rangle$ and $Q_{nm} = \langle n \,|\, Q \,|\, m \rangle$. According to the **GR** theory, the luminosity L_{GW}, in the inspiral stage, is given by the **quadrupolar radiation:**[11-14]

$$L_{GW} = (32\mu^2 G/5c^5) r^4\omega^6 = (8G\omega^6/5c^5)\, M^2 r^4 = (8G\omega^6/5c^5) Q^2 \qquad (\Phi.4),$$

where $Q = Mr^2$ is the mass quadrupole of the **μBHb**. Thus, by analogy with the predicted electromagnetic radiation and based in the *Correspondence Principle* we could believe that the QM gravitational luminosity $(L_{GW})_{nm}$ can be estimated by

$$(L_{GW})_{nm} = \hbar\omega P^{+}_{nm} \approx (8G\omega^6/5c^5)\,|<n\,|\,Q\,|\,m>|^2 \qquad (\Phi.5).$$

In **Appendix D** is shown a different approach of Weinberg[15] to calculate $(L_{GW})_{nm}$.

Now**,** let us give a reasonable justification for **Eq.(Φ.5).** Thus, let us assume that W^+ (t) is proportional to the small perturbations $h_{\mu\upsilon}$ of the tensor metric $g_{\mu\upsilon}$ created by the quadrupole temporal oscillations $Q_{\alpha\beta}(t)$[16,19] of the μBBH that are written as

$$Q_{xx}(t) = 3\mu r^2[1 + \cos(2\omega t)]/2 \quad \text{and} \quad Q_{yy}(t) = 3\mu r^2[1- \cos(2\omega t)]/2 \qquad (\Phi.6).$$

where $\mu = m_1m_2/(m_1 + m_2)$ and ω is the orbital angular frequency (see **Appendix A).** That is, $g_{\mu\nu}$ is slightly modified, $g_{\mu\upsilon} \approx g_{\mu\upsilon}^{(o)} + h_{\mu\upsilon}$, where $h_{\mu\upsilon}$ is due to quadrupolar effects pointed above. Taking into account that [14,19] $h_{\alpha\beta}(t,\mathbf{x}) = (2G/c^2r)(\partial^2Q_{\alpha\beta}/\partial t^2)$ the "classical" gravitational luminosity L_{GW} is given by (see **Appendix A**)

$$L_{GW} = (G/45c^5) < (\partial^3Q_{\alpha\beta}/\partial t^3)^2 > = (G/45c^5)\,[< (\partial^3Q_{xx}/\partial t^3)^2 > + [< (\partial^3Q_{yy}/\partial t^3)^2 >] =$$

$$= (32\mu^2G/5c^5)r^4\omega^6 = (8G\omega^6/5c^5)Q^2 \qquad (\Phi.7),$$

where $Q = \mu r^2$ is the μBBH *mass* quadrupole. So, putting $(L_{GW})_{nm} = \hbar\omega P^{+}_{nm}$,we have $w^+(t) \sim h_{\alpha\beta}(t)$ and using **Eq.(Φ.2)** we will assume that the QM the *gravitational luminosity* $(L_{GW})_{nm}$ can be estimated by

$$(L_{GW})_{nm} = \hbar\omega P^{+}_{nm} \approx (8G\omega^6/5c^5)\,|<n\,|\,Q\,|\,m>|^2 \qquad (\Phi.8),$$

in agreement with **Eq.(Φ.5).** At this point it is important to analyze this proposed mechanism to explain the decay transitions in mBBH. Indeed, as seen in **Appendix A**, the amplitude of the emitted GW are given by $\Psi_{\alpha\beta}(t,\mathbf{x}) = h_{\alpha\beta}(t,\mathbf{x}) = (2G/c^2R)(\partial^2Q_{\alpha\beta}/\partial t^2)$. That is, GW are emitted due to the "*metric perturbation*" $h_{\alpha\beta}(t)$. To obtain **Eq.(Φ.5)** a similar hypothesis is assumed: the time dependent metric modification is responsible by a potential interaction W^+ that induces transitions n → m between quantum states. The gravitational luminosity would now be given by $(L_{GW})_{nm} = \hbar\omega P^{+}_{nm}$. That is, gravitational quantum transitions are induced by metric perturbations due to mass quadrupolar effects. In the electromagnetic quantum field theory transitions are induced by "vacuum" fluctuations due to electric quadrupoles.

Evaluation of the spiral time.

To evaluate the QM "spiral time" τ we must remember that in this stage, according to **Eqs.(Φ.8)** and **(3.12)** the energy levels $E^g_n = -\Theta_{grav}/n^2$ are very close since quantum numbers are very large, e.g. $n > 10^{24}$. As there is a "continuum of levels" it is expected, according to the CP, the mBBH description given by quantum mechanics approaches asymptotically a state of motion obtained with the "classical" GR. Indeed, for the inspiral stage **Eq.(Φ.8)** can be written as

$$(L_{GW})_{ab} = (dE/dt)_{ab} \approx (8G\omega^6/5c^5)\, M^2 r^4 = (8\, M^2 G\omega^6/5c^5)\, r^4 \qquad (\Phi.10).$$

which is similar to **Eq.(2.1)** given by the "classical" GR.

(4)Conclusions and Discussions.

(4.1)A reasonable agreement between the estimated luminosity and inspiral time is obtained with the GR and the quantum approach. So, it seems that in the mBHb spiral motion the effects of the gravitation interaction can be quantized in a non relativistic limit of Schrödinger - Newton equation.

(4.2) As, in the inspiral motion, according to **Appendix C**, quantum states $|a\rangle$ and $|b\rangle$ of the mBBH are represented by $u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\, |\ell m\rangle$ the quadrupole matrix elements are written as

$$Q_{ab} = \int dr\, r^4\, R_a(r)\, R_b(r) \langle \ell_b m_b | Y_{2m}^*(\theta,\varphi) | \ell_a m_a \rangle \qquad (4.2.1).$$

Eq.(4.2.1) shows that, according to the Wigner-Eckart Theorem,[4] quadrupole **t**ransitions $a \rightarrow b$ are allowed only if $\ell_b = \ell_a \pm 2$ and $m_b = m_a + 2$. So, if GW are composed by "gravitons", selection rules dictated by the matrix elements in Eq.(4.2.1) ***suggest*** that "gravitons" have spin 2.

(4.3) According to **Appendix (A.1)** the gravitational luminosity L_{GW} emitted by a BH binary with black holes with equal mass μ, is given by $L_{GW} = (32\mu^2 G/5c^5) r^4 \omega^6$. In the radiation zone the gravitational energy is transported by a **plane wave** with amplitude $h(\omega)$ given by Eq.(A.17)[13,14]

$$h(\omega) = (4^{2/3}/\sqrt{36})\, [(G\mu)^{5/3}/Rc^4]\, \omega^{2/3} \qquad (4.3.1)$$

where **R** is distance from the **BHb** and the observer at the radiation zone.

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Appendix A. Emission of gravitational waves by BHb.

In GR [5,14-16], assuming that the gravitation field is *weak* and that the bodies have small velocities compared with the light velocity, the space-time metric tensor $g_{\mu\upsilon}$ we can put $g_{\mu\upsilon} \approx g_{\mu\upsilon}^{(o)} + h_{\mu\upsilon}$, where $h_{\mu\upsilon}$ is as mall perturbation of $g_{\mu\upsilon}^{(o)}$.[5,14-16] In the Newtonian limit we have $g_{oo} = -1 - 2\varphi/c^2$, where $\varphi = GM/r$.[5] In these conditions the Ricci tensor R_{ik} can be written as

$$R_{ik} = -(1/2)\Box h_{\mu\upsilon} \quad \text{(A.1).}$$

Defining the gravitational field as $\Psi_{\mu\upsilon} = h_{\mu\upsilon} - (1/2)\delta_{\mu\upsilon} h$, where $h = h_{\alpha}{}^{\alpha}$, in *weak field* limit the field $\Psi_{\mu\upsilon}$ obeys the equations[5, 14-16]

$$\Box\Psi_{\mu\upsilon} = -(16\pi G/c^4)\tau_{\mu\upsilon} \quad \text{and} \quad \partial_{\mu}\Psi^{\mu\upsilon} = 0 \quad (\textit{gauge condition}) \quad \text{(A.2),}$$

where $\tau_{\mu\upsilon}$ is a pseudo-tensor mass-energy momentum.

The solution of **Eq.(A.2)** for retarded times is given by[5,18]

$$\Psi_{\mu\upsilon}(\mathbf{x},t) = -(4G/c^4) \int \tau_{\mu\upsilon}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x})\, d^3\mathbf{x}' / |\mathbf{x} - \mathbf{x}'| \quad \text{(A.3),}$$

where the integration is over the volume V of the system.

Supposing that gravitational effects are observed very far from the origin O ("wave zone") where they are produced, that is, $|\mathbf{x}| = R >> |\mathbf{x}'|$ we get from **Eq.(A.3),** remembering that we have a retarded time function $\tau_{\mu\upsilon}$:

$$\Psi_{\mu\upsilon}(x,t) \approx -(4G/c^4R) \int \tau_{\mu\upsilon}\, d^3\mathbf{x}' \quad \text{(A.4).}$$

Integrating Eq.(A.4) over the volume V we obtain the gravitational field[5,13]

$$\Psi_{\alpha\beta}(x,t) = (2G/c^2R)\,(\partial^2 Q_{\alpha\beta}/\partial t^2) \quad \text{(A.5),}$$

where $Q_{\alpha\beta}$ is the *mass quadrupole moment* of the emitting system defined by

$$Q_{\alpha\beta} = \int \rho_o(\mathbf{x}')(3x'_{\alpha}x'_{\beta} - r'^2\delta_{\alpha\beta})\, d^3\mathbf{x}'$$

where ρ_o is the mass density. At this point it opportune to remember that gravitational multipoles are defined by the potential expansion [14]

$$\varphi(\mathbf{x}) = -G \int \rho_o(\mathbf{x}')d^3\mathbf{x}'/|\mathbf{x} - \mathbf{x}'| \approx -Gm/r - (G/r^3)\,\mathbf{x}.\mathbf{D} - (G/2r^5)\Sigma_{\alpha\beta}\, Q^{\alpha\beta}\, x^{\alpha}x^{\beta} + ... \quad \text{(A.6),}$$

where $m = \int \rho_o(\mathbf{x}')\, d^3\mathbf{x}'$, $\mathbf{D} = \int \rho_o(\mathbf{x}')\, \mathbf{x}' d^3\mathbf{x}'$ and $Q_{\alpha\beta} = \int \rho_o(\mathbf{x}')(3x'_{\alpha}x'_{\beta} - r'^2\delta_{\alpha\beta})\, d^3\mathbf{x}'$.

The *mass dipole moment* is null ($\mathbf{D} = 0$) since the origin of coordinates O is chosen to coincide with the center of mass.

In vacuum we have the *traditional wave equations*

$$\Box\Psi_{\mu\upsilon} = \Box h_{\mu\upsilon} = 0 \quad \text{with the "gauge "} \quad \partial(h^{\mu}{}_{\nu})/\partial x^{\mu} = 0 \quad \text{(A.7)}$$

showing that the gravitational field propagates with the light velocity. Note that the tensor field $h_{\mu\upsilon}$ is obtained integrating **Eq.(A.4)** as will be seen later.

At this point we find a fruitful analogy with the electromagnetism. The Maxwell equations in *Lorentz gauge* in empty space are $\Box A_{\mu} = 0$ and $\partial A^{\mu}/\partial x^{\mu} = 0$.

Let us consider a plane GW, that is, a field that changes only in one direction z of the space. Choosing z > 0 as the direction of propagation of the wave we can write $h_{ik} = h_{ik}(t - z/c)$. So, the wave equation **Eq.(A.7)** becomes

$$[\partial^2/\partial z^2 - (1/c^2)(\partial^2/\partial t^2)]\, h_{ik} = 0 \qquad (A.8)$$

that has the familiar solution with the gauge condition,

$$h_{ik}(z,t) = A_{ik} \cos(k_\mu x_\mu) \qquad (A.9),$$

where $k_\mu = (0,0,k,\omega)$, $k = k_z = |\mathbf{k}| = \omega/c$ is the wave vector and ω is the frequency of the wave. As $h_{ik}(z)$ obey (A.8) the following conditions are obeyed: $A_{\beta\alpha}k^\alpha = 0$ and $k_\alpha k^\alpha = 0$. Under these conditions the **amplitude tensor A_{ik}** has only 4 non-null components $A_{11} = -A_{22}$, $A_{12} = A_{21}$ with the condition $Tr(A_{ik}) = A_i^{\,i} = 0$ and only the following **transversal components** to the z-direction of propagation: $A_{xx} = -A_{yy}$ and $A_{xy} = A_{yx}$.

$$A_{ik} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The transversal fields h_{xx}, h_{yy} and h_{xy} are represented using (2x2) matrices called polarization matrices $(\varepsilon_+)_{ik}$ and $(\varepsilon_x)_{ik}$:

$$(\varepsilon_+)_{ik} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad (\varepsilon_x)_{ik} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (A.10)$$

The general solution of **Eq.(A.8)** can be written as a linear combination of the fields h_{ik}, with polarizations (+) and (x), respectively:

$$h_{ik}^{(+)} = h_+ (\varepsilon_+)_{ik} \cos(\omega t - kz) \quad \text{and} \quad h_{ik}^{(x)} = h_x (\varepsilon_+)_{ik} \cos(\omega t - kz + \alpha) \qquad (A.11),$$

where $h_+ = A_{11}$, $h_x = A_{12}$ and α is an arbitrary phase. The tensorial polarization of the GW creates an effect much more complicate than the linear polarization of the electromagnetic waves. These fields deform the space-time creating tidal (shear) on the matter . The line forces due to the polarizations (X) and (+) are shown in Figure 2.

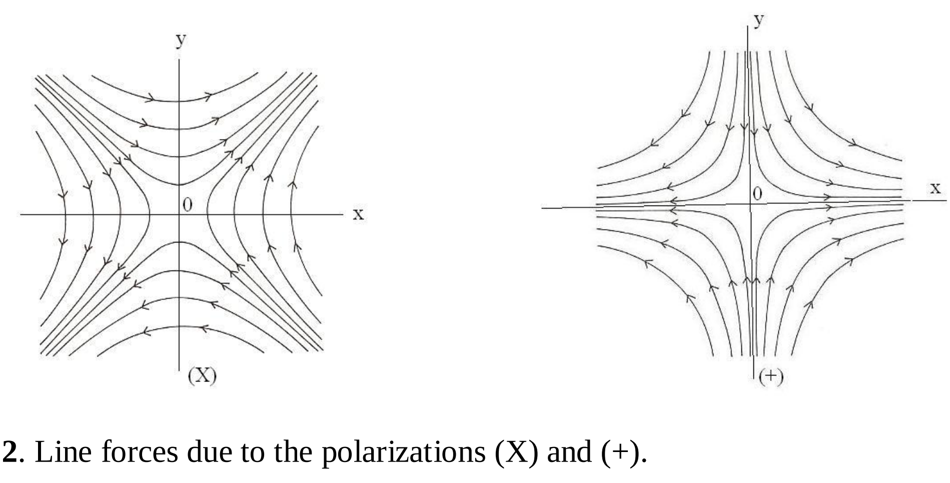

Figure 2. Line forces due to the polarizations (X) and (+).

The total energy emitted per unit of time dE/dt or "gravitational luminosity" L_{GW} is given by[5,14]

$$L_{GW} = dE/dt = -(G/45c^5) < (\partial^3 Q_{\alpha\beta}/\partial t^3)^2 > \qquad (A.12),$$

where the brackets indicates a time average and are taken into account the effect of all components of the quadrupole tensor. Note that the GW is a tensor function not a scalar function like an electromagnetic wave.

(A.1)GW emitted by BHb.

For a binary system (see **Fig.1**) composed by stars with masses m_1 and m_2 separated by a distance r one can show[14,19] that

$$Q_{xx} = 3\mu r^2[1 + \cos(2\omega t)]/2 \quad \text{and} \quad Q_{yy} = 3\mu r^2[1- \cos(2\omega t)]/2 \qquad (A.13),$$

where $\mu = m_1 m_2/(m_1 + m_2)$ and ω is the **orbital angular frequency.** In these conditions one see that $h_{\alpha\beta}(t,\mathbf{x})$, using **Eqs.(A.11)** and **(A.13),** would be given by

$$\Psi_{\alpha\beta}(t,\mathbf{x}) = h_{\alpha\beta}(t,\mathbf{x}) = (2G/c^2R)(\partial^2 Q_{\alpha\beta}/\partial t^2) \sim h \cos(2\omega t) \qquad (A.14),$$

where $h = 6\mu G r^2/Rc^2$. Showing that the GW frequency is $\boldsymbol{\omega_g = 2\omega}$.

Using **Eqs.(A.12)** and **(A.13)** we obtain

$$L_{GW} = (G/45c^5) < (\partial^3 Q_{\alpha\beta}/\partial t^3)^2 > = (G/45c^5) [<(\partial^3 Q_{xx}/\partial t^3)^2 > + [<(\partial^3 Q_{yy}/\partial t^3)^2 >] =$$

$$= (32\mu^2 G/5c^5) r^4 \omega^6 \qquad (A.15).$$

As the energy of the GW in the radiation zone is transported by a **plane wave** with amplitude h and rotation frequency ω one can show that[13,14]

$$h^2 = (8\pi G/\omega^2 c^3)\,(L_{GW}/4\pi R^2) \qquad (A.16).$$

As Kepler´s law for a binary [1,5] says that $\omega^2 r^3 = G(M_1 + M_2)$ and $M_1 = M_2 = M^*$ we get $r = (2GM^*/\omega^2)^{1/3}$. Substituting this r value in **Eq.(A.16)** we obtain h as a function of the orbital angular frequency ω (rad/s):[11,12]

$$h(\omega) = (4GM^*/Rc^4\sqrt{36})(2GM^*/\omega^2)^{2/3}\omega^2 = (4^{2/3}/\sqrt{36})\,[(GM^*)^{5/3}/Rc^4]\,\omega^{2/3} \qquad (A.17).$$

Recently [11-13] gravitational waves have been detected, with frequencies $\omega \sim 160\,\pi$ rad/s. They have been emitted by a black hole binary (BHb). The BHb, that was distant $R \sim 1.3\ 10^9$ light years $\sim 1.2\ 10^{25}$ m from the Earth had $M^* \sim 20$ solar masses. Using **Eq.(A.17)** and taking into account the BHb parameters given above we see that

$$h(\omega) \sim 10^{-23}\,\omega^{2/3} \qquad (A.18).$$

The measured average amplitude $< h >$ for frequencies $\omega \sim 160\,\pi$ rad/s was found to be $< h > \sim 10^{-21}$, in good agreement with the experimental results.

Appendix B. Classical electromagnetic radiation.

According to classical Electrodynamics[2]

$$\Box \mathbf{A}(\mathbf{x},t) = -\mu_o \mathbf{J}(\mathbf{x},t) \qquad \text{(B.1)},$$

where $\Box$ is the d´Alembertian operator $\Box = \partial_\mu \partial^\mu$. The solution of (A.1)is given by[2]

$$\mathbf{A}(\mathbf{x},t) = \mu_o \int d^3\mathbf{x}' \int dt' [\mathbf{J}(\mathbf{x}',t')/|\mathbf{x} - \mathbf{x}'|]\ \delta\ (t' + |\mathbf{x} - \mathbf{x}'|/c - t) \qquad \text{(B.2)}.$$

With the sinusoidal time dependence $\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x}) \exp(-i\omega t)$ (A.1) becomes given by

$$\mathbf{A}(\mathbf{x},t) = \mu_o \int \mathbf{J}(\mathbf{x}') \exp(ik|\mathbf{x} - \mathbf{x}'|)/|\mathbf{x} - \mathbf{x}'|\ d^3\mathbf{x}' \qquad \text{(B.3)},$$

that can be expanded in series taking into account that the fields are very far from the source, that is, $r >> d$ and that $d << \lambda$, where d is the dimension of the source and λ the wavelength of the emitted radiation. The rate of the emitted electromagnetic radiation dE/dt can be calculated expanding $\mathbf{A}(\mathbf{x},t)$ using *electric and magnetic multipoles.* [2]

In vacuum (A.1) obeys the equation

$$\Box \mathbf{A}(\mathbf{x},t) = 0 \qquad \text{(B.4)}.$$

The general solutions of the above equations for **A** is formed by superposing transverse waves[2] of the field $\mathbf{A}(x_\mu)$. In *second quantization* context [4,21] planes waves **A** are written as (omitting details of normalization constant, wave polarization,...) where $k_\mu = (\mathbf{k}, i\omega/c)$,

$$\mathbf{A}(x_\mu) = \sum_{k\omega} [\mathbf{a}_{k\omega} \exp(ik_\mu x_\mu) + \mathbf{a}^*_{k\omega} \exp(-ik_\mu x_\mu)] \qquad \text{(B.5)},$$

(B.1) Emitted electromagnetic energy per unitof time dE/dt.

If the emitted radiation is mainly due to the electric dipole $\mathbf{D} = \int \mathbf{x}' \rho_e(\mathbf{x}')\ d^3\mathbf{x}'$ we have [2]

$$dE/dt = (ck^4/3)\ |\mathbf{D}|^2 = (\omega^4/3c^3)\ |\mathbf{D}|^2 \qquad \text{(B.6)},$$

where $\rho_e(\mathbf{x}')$ is the electric charge density and $k = 2\pi/\lambda = \omega/c$.

If the energy is mainly emitted by electric quadrupole $Q_{\alpha\beta}$ and by magnetic dipole **m** we can show that **[2]**

$$dE/dt = (ck^6/360) \sum_{\alpha\beta} |Q_{\alpha\beta}|^2 \qquad \text{(B.7)},$$

where $Q_{\alpha\beta} = \int \rho_e(\mathbf{x}')(3x'_\alpha x'_\beta - r'^2 \delta_{\alpha\beta})\ d^3\mathbf{x}'$ and $\mathbf{m} = \int \mathbf{x}' \mathrm{x}\ \mathbf{J}(\mathbf{x}')\ d^3\mathbf{x}'$.

(B.2)Larmor Acceleration Formula.

According to the classical electrodynamics accelerated charges emit radiation and the dominant energy loss is from electric dipole which obeys the Larmor formula (in Gaussian units),[2,17]

$$dE/dt = (2/3c^3)|d^2\mathbf{D}/dt^2| \qquad \text{(B.8)}.$$

This formula can be used to estimate the classical lifetime of the **Bohr atom**.[17] For very large quantum numbers n, Bohr's *correspondence principle* **(CP)** demands that classical physics and quantum physics give the same answer, at least in average. In these conditions as the energy levels are very close the radiate energy is estimated using the classical electrodynamics.[17] So, putting **D** = e**r** it is assumed that the electron moves in circular orbits around the nucleus emits continuously radiating energy according to,

$$dE/dt = (2/3c^3)e^2\mathbf{a}(t)^2 \qquad (B.9),$$

where **a** the electron acceleration, which is essentially the radial one $a_r = r\omega^2$. In this *adiabatic approximation* the electronic orbit remains nearly circular at all times whith $\omega \approx$ constant. According to reference [17] the electron will fall to the origin, following a spiral motion, after a time $t_{fall} \sim 10^{-11}$ s. The observed lifetime of the $2p^{1/2}$ state of the hydrogen is $\sim 10^{-9}$ s (see **Appendix C**). In quantum mechanics the ground state, however, "appears" to have infinite lifetime. The accelerated electron along a radius r(t) with a tangential speed $v_\Theta(t)$ and angular speed $\omega = d\Theta/dt = v_\Theta(t)/r$ emits a wave with frequency ω called *synchrotron radiation*.

Taking into account that $|a| \sim a_r = r\omega^2$ **Eq.(B.9)** becomes written as

$$dE/dt \approx (2e^2\omega^4/3c^3)\, \mathbf{r}(t)^2 \qquad (B.10).$$

Appendix C. Quantum electromagnetic radiation.

In Special Relativity (**SR**) [2,4] the generalized vector potential is defined by $A_\mu = (\mathbf{A}, iA_o) = (\mathbf{A}, i\varphi)$. A free particle with a mass m has a 4-momentum $p_\mu = (\mathbf{p}, iE)$ where E is the total energy $E = (m^2c^2 + p^2c^2)^{1/2}$. The 4-momentum a charged particle submitted to an electromagnetic field becomes given by $p_\mu \rightarrow p_\mu - (e/c)\, A_\mu$. That is, $E \rightarrow E - e\varphi$ and $\mathbf{p} \rightarrow \mathbf{p} - (e/c)\mathbf{A}$.

The relativistic wave equation [4] for a charged spin zero particle submitted to an external electromagnetic field is obtained through the transformation

$$p_\mu - (e/c)\, A_\mu \rightarrow - i\hbar\, \partial/\partial_{x\mu} - (e/c)\, A_\mu \qquad (C.1),$$

that is

$$\{ \Sigma_\mu (- i\hbar\, \partial/\partial_{x\mu} - (e/c)\, A_\mu)^2 + m^2c^2 \}\Psi = 0 \qquad (C.2),$$

or

$$(1/c^2)[i\hbar\, \partial/\partial t - e\varphi]^2\, \Psi = [(i\hbar\, \mathrm{grad} - (e/c)\mathbf{A})^2 + m^2c^2\,]\Psi \qquad (C.3).$$

According to quantum mechanics[4] the interaction of a charged spinless particle with the electromagnetic radiation is given by the operator, putting $\boldsymbol{p} = -i\hbar$ grad,

$$W(t) = -(e/mc)(\mathbf{A}.\boldsymbol{p}) + (e^2/2mc^2)\mathbf{A}^2 \qquad (C.4),$$

where the vector potential **A** is written in the form of a plane wave with wave vector **k** and frequency ω, $\mathbf{A}(\mathbf{r},t) = A_o\, \mathbf{u} \cos[\mathbf{k}.\mathbf{r} - \omega t]$, with **u** the unit vector determining the polarization of the radiation (direction of the electric field vector). With the perturbation theory to evaluate the transitions probabilities, in a first order approximation, we neglect the term $(e^2/2mc^2)\mathbf{A}^2$ since it is gives a small contribution, of the order of $\alpha = e^2/hc \sim 1/137$.[4] In this way we retain only the first term of (C.4),

$$W(t) = -(e/mc)(\mathbf{A}.\boldsymbol{p}) \qquad \text{(C.5).}$$

The amplitude a_o will be determined in such a way that there are an average N photons of energy ħω and polarization **u** in a volume V . So, from

$$\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t = A_o\,\mathbf{u}\,(\omega/c)\sin[\mathbf{k.r} - \omega t] \qquad \text{and}$$

from the condition

$$N\hbar\omega/V = \langle \mathbf{E}^2(t)\rangle/4\pi = (A_o^2\omega^2/4\pi c^2)\langle \sin^2[\mathbf{k.r} - \omega t]\rangle = A_o^2\omega^2/8\pi c^2$$

we see that $A_o = 2c(2\pi\hbar N/\omega V)^{1/2}$.

Writing $W(t) = w\exp(i\omega t) + w^*\exp(-i\omega t)$ where $w = A_o\exp(-i\mathbf{k.r})(\mathbf{u}.\boldsymbol{p})$ the transition probability per unit of time for a transition from a (initial) state $|b\rangle$ to a (final)state $|a\rangle$ with the *emission* of a quantum ħω will be determined by the expression

$$P_{ab} = (2\pi/\hbar)\,|\langle a|w|b\rangle|^2\,\rho(E_{fin}) \qquad \text{(C.6),}$$

where the initial energy E_{init} = final energy E_{fin} or $E_a = E_b + \hbar\omega$ and $\rho(E_{fin}) = \rho(\hbar\omega)$ [4] is the density of final photon states $dN/d\varepsilon = \rho(\hbar\omega) = [V\omega^2/(2\pi c)^3\hbar]d\Omega$, remembering that for photons $\varepsilon = \hbar\omega$ and $p = \varepsilon/c$. The matrix element $\langle a|w|b\rangle$ is given by

$$\langle a|w|b\rangle = -A_o\langle a|\,e^{-i\mathbf{k.r}}(\mathbf{u}.\boldsymbol{p})\,|b\rangle \qquad \text{(C.7),}$$

remembering that $\boldsymbol{p} = -i\hbar$ grad. Since the integration of matrix element is will be essentially over the region (**r**) of the size (a) of emitting system it is convenient to expand the exponential factor in a power series,

$$e^{-i\mathbf{k.r}} = 1 - i(\mathbf{k.r}) + [-i(\mathbf{k.r})]^2/2! + ... = \qquad \text{(C.8).}$$

(C.1) Dipole radiation.

When $ka = 2\pi/\lambda \ll 1$, where λ is the wavelength of the emitted photon, it is enough to consider only of the first term of **Eq.(C.8)** obtaining:[4]

$$\langle a|w|b\rangle = -i\,\omega_{ab}\,A_o(\mathbf{u}.\boldsymbol{D})_{ab} \qquad \text{(C.9),}$$

where $\boldsymbol{D} = \Sigma_i q_i \boldsymbol{r}_i$ is the ***electric dipole moment operator*** of the emitting system with discrete charges q_i. One can show that

$$\langle a|w|b\rangle = -i\,\omega_{ab}\,A_o\,\mathbf{u}.(\boldsymbol{D}_{ab}) \qquad \text{(C.10),}$$

where the vector $\boldsymbol{D}_{ab} = \langle a|\mathbf{D}|b\rangle$ is called the *electrical dipole moment of the b → a transition.* In this way, using **Eqs.(C.6)-(C.10)** we get the probability per unit of time dP_{ab}^+ that a photon with polarization **u** and frequency $\omega = |\omega_{ab}| = (E_a - E_b)/\hbar$ is emitted within a solid angle dΩ ,

$$(dP_{ab}^+)_{dip} = N(\omega^3/2\pi\hbar c^3)\,|\mathbf{u}.(\boldsymbol{D}_{ab})|^2\,d\Omega \qquad \text{(C.11).}$$

The polarization **u** is perpendicular to the direction of propagation **k.** If we denote by θ the angle between **k** and the dipole moment of the transition $\boldsymbol{D}_{ab}$ we have $|\mathbf{u}.(\boldsymbol{D}_{ab})|^2 = |\boldsymbol{D}_{ab}|^2 \sin^2\theta$. Thus,

$$(dP_{ab}^{+})_{dip} = N\,(\omega^3/2\pi\hbar c^3)\,|\boldsymbol{D}_{ab}|^2 \sin^2\theta\, d\Omega \qquad \text{(C.12).}$$

Integrating Eq.(C.12) with N =1[4] over all directions of the radiation we get the *total transition probability per unit of time* P_{ab} involving the *emission of one photon*:

$$(P_{ab}^{+})_{dip} = (4\omega^3/3\hbar c^3)\,|\boldsymbol{D}_{ab}|^2 \qquad \text{(C.13).}$$

To estimate the order of magnitude of Eq.(C.13) for atomic systems with linear dimension a we put $\boldsymbol{D} = e\boldsymbol{r}$ taking $|\boldsymbol{r}_{ab}| = a \approx e^2/\hbar\omega$. Thus, $(P_{ab}^{+})_{dip}$ can be written as

$$(P_{ab}^{+})_{dip} \approx (e^2\omega/\hbar c)(\omega a/c)^2 \approx \omega/(137)^3,$$

that for optical radiation ($\omega \sim 10^{15}$/s) gives $(P_{ab})_{dip} \sim 10^9$/s. The observed lifetime $\tau \sim 1/(P_{ab})_{dip}$ of the $2p^{1/2}$ state of the hydrogen is $\tau \sim 10^{-9}$ s.[4]

Consequently, energy emitted per unit of time dE/dt will be given by $(dE_{ab})_{dip} = \hbar\omega(P_{ab}^{+})_{dip}$, that is,

$$(dE/dt)_{dip} = (4\omega^4/3c^3)\,|\boldsymbol{D}_{ab}|^2 \qquad \text{(C.14).}$$

In case of the Bohr atom with $\mathbf{D} = e\mathbf{r}$ (C.14) becomes written as

$$(dE/dt)_{dip} = (4e^2\omega^4/3c^3)\,|\boldsymbol{r}_{ab}|^2 \qquad \text{(C.15).}$$

It becomes equal to Eq.(B.8) if the average energy (averaged over the time) emitted per unit of time is due to a dipole $\mathbf{D}(t) = e\mathbf{r}(t) = 2\,(|\boldsymbol{D}_{ab}|^2)^{1/2}\cos(\omega t) = 2e\,|\boldsymbol{r}_{ab}|\cos(\omega t)$.

(C.2)Quadrupole radiation.

If it is necessary to take into account the second term of the expansion (B.8) the matrix element $< a\,|w|\,b >$ given by **Eq.(C.7)** will be

$$< a\,|w|\,b > \;= -i\,A_o < b\,|(\mathbf{k}.\mathbf{r}')(\mathbf{u}.\mathbf{p}')|\,a > \;= A_o\,(\hbar k/2)\mu\omega < b|\,r'(\mathbf{n}.\mathbf{r}')|\,a > \qquad \text{(C.16),}$$

where $\omega_{ab} = \omega$, μ the electron mass and $\mathbf{n} = \mathbf{r}'/r'$. Eq.(C.16) would be responsible for *electric quadrupole* transitions involving matrix elements of the products xy, xz and yz and *dipole magnetic* transitions of matrix elements of the angular momentum operators L_x, L_y and L_z. In quantum systems with spherically symmetric potential magnetic dipole transitions give no contributions to photons emission.[4] So, following the same procedure used for dipole radiation we can calculate the total emission probability per unit of time within the solid angle dΩ. The general angular distribution of the quadrupole radiation is very complicated. [2,20,21] As we only intend to obtain an order of magnitude of the quadrupole radiation we put

$$(P_{ab}^{+})_Q \approx (\omega^5/2\pi\hbar c^5)|Q_{ab}|^2 \qquad \text{(C.17),}$$

where, the quadrupole matrix element is represented by Q_{ab}. So, the total energy per unit of time $(dE/dt)_Q$ emitted by the quadrupole is given by

$$(dE/dt)_Q \approx (\omega^6/2\pi c^5)|Q_{ab}|^2 \qquad \text{(C.18).}$$

In classical electrodynamics we have[2]

$$(dE/dt)_{class} \approx (ck^6/240)Q_o^2 = (\omega^6/240c^5)\, Q_o^2 \qquad \text{(C.19).}$$

Let us estimate $(P_{ab}^+)_Q$, given by **Eq.(C.17),** for systems emitting optical frequencies $\omega \sim 10^{15}$/s and with atomic dimensions a ~10^{-7} cm. Taking $Q_{ab} \sim ea^2$ we verify that

$$(P_{ab}^+)_Q \approx (\omega^5/2\pi\hbar c^5)|Q_{ab}|^2 \sim 10^5/s \qquad \text{(C.20),}$$

that is, $(P_{ab}^+)_Q \sim 10^{-4}\,(P_{ab}^+)_{dip}$.

(C.3)Multipole tensor operators $T_{\ell m}(\theta,\varphi)$.

Since calculations of quadrupole and magnetic dipole transitions and of higher order terms of the expansion (B.8) are very intricate it is convenient to use a different approach to estimate these matrix elements. In this way are used the *tensor multipole operators* $T_{\ell m}(\theta,\varphi)$ defined by [2,4,20,21]

$$T_{\ell m}(r,\theta,\varphi) = [4\pi/(2\ell+1)]^{1/2}\, r^\ell\, Y_{\ell m}(\theta,\varphi) = [4\pi/(2\ell+1)]^{1/2}\, r^\ell\, |\ell m\rangle \qquad \text{(C.21),}$$

where ℓ =1, 2,...correspond to dipole, quadrupole ,... and the angle θ is between **k** and **r.**

If the state functions are given by $u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\,|\ell m\rangle$ the transition probabilities per unit of time P_{ab} will directly proportional to $|a_E(\ell,m)|^2$ where the amplitudes $a_E(\ell,m)$ are given, for ka << 1, by[4]

$$a_E(\ell,m) = -\,[4\pi/(2\ell+1)!!](\ell+1/\ell)^{1/2}\, k^{\ell+2}\, Q_{\ell m} \qquad \text{(C.22),}$$

where

$$Q_{\ell m} = \int dr\, r^{\ell+2}\, R_a(r)\, R_b(r) < \ell_b m_b|Y_{\ell m}^*(\theta,\varphi)|\,\ell_a m_a >.$$

The matrix element $< n'j'm'|T_k^q|\, n\, j\, m >$ according to the Wigner-Eckart Theorem (WET)[22] is given by $< n'j'm'|T_k^q|\, n\, j\, m > = (jkmq|j'm')\,(n'j'||Tk||nj)$, where $(jkmq|j'm') \neq 0$ only when $m + q = m'$ and $|j - k| \leq j' \leq j + k$.

For **dipole (ℓ=1)** using Eq.(C.18) the transition probabilities per unit of time P_{ab} between states | a > and | b > are proportional to $|\mathbf{D}_{ab}|^2$ where,

$$|\mathbf{D}_{ab}| = (4\pi/3)^{1/2} \int dr\, r^3\, R_a(r)\, R_b(r) < \ell_b m_b|Y_{10}(\theta,\varphi)|\,\ell_a m_a > \qquad \text{(C.23).}$$

Thus, following the WET the a → b transition is allowed only if we have:

$$\ell_b = \ell_a \pm 1 \quad \text{and} \quad m_b = m_a\,.$$

This kind radiation is called *electrical dipole radiation* and is denoted by E1.

For electric **quadrupole (ℓ=2)** P_{ab} is proportional to $|Q_{ab}|^2$ where

$$Q_{ab} = \int dr\, r^4\, R_a(r)\, R_b(r) < \ell_b m_b|Y_{2m}^*(\theta,\varphi)|\,\ell_a m_a > \qquad \text{(C.24),}$$

showing that quadrupole **t**ransitions a → b are allowed only if

$$\ell_b = \ell_a \pm 2 \quad \text{and} \quad m_b = m_a + 2 \qquad \text{(C.25).}$$

This kind of radiation is called *electric quadrupole radiation* and is denoted by E2.

(C.4)Second quantization approach.

Basic ideas on the quantization of radiation can be seen in many books. In *vacuum*, with the Lorentz gauge the electromagnetic field $\mathbf{A}(x^\mu)$ is given by[4,21]

div($\mathbf{A}$) = 0, $\partial_\mu\partial^\mu = \Box\mathbf{A} = 0$, μ =1,2,3,4, $x_\mu = (\mathbf{x}, ict)$ and $A_\mu = (\mathbf{A}, i\varphi)$.

The general solutions of the above equations for $\mathbf{A}$ is formed by superposing transverse waves[2,4] of the field $\mathbf{A}(x_\mu)$. In the *second quantization* context planes waves $\mathbf{A}$ are written as (omitting details of normalization constant, wave polarization,...)

$$\boldsymbol{A}(x_\mu) = \Sigma_{k\omega} [\mathbf{a}_{k\omega} \exp(ik_\mu x_\mu + \mathbf{a}^*_{k\omega} \exp(-ik_\mu x_\mu)]/\sqrt{\omega} \qquad \text{(C.26)},$$

where $k_\mu = (\mathbf{k}, i\omega/c)$, $\mathbf{a}_{k\omega}$ and $\mathbf{a}^*_{k\omega}$ are the **creation and annihilation photon operators**, respectively.

In this approach transition probabilities P_{ab} are now estimated with **Eq.(C.26)** the field operator **A** defined by **Eq.(C.22).** Taking into account transitions involving *vacuum states* and wavefunctions $u_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)|\ell m>$ we get the same results obtained before without the second quantization approach. The main difference now is that the electromagnetic radiation is composed by *photons*. Selection rules obeyed in *electrical dipole radiation* (E1) show that **photons** must have spin 1.

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